Landau Damping

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What is a Plasma?

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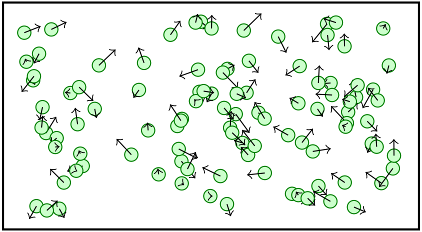
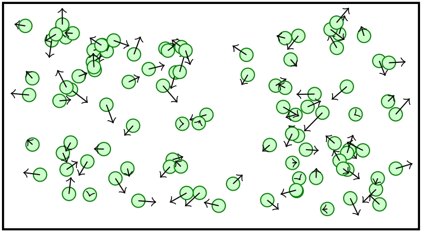
A plasma is a gas with ionized particles (not necessarily all)



Here we consider freely moving electrons within a fixed background potential (ions are much heavier)

Level of Description



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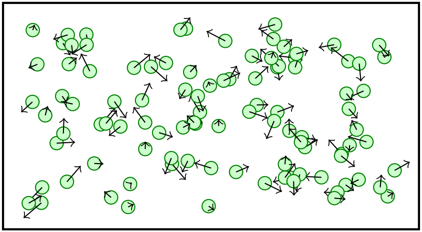

- Microscopic viewpoint
  Use newtonian equations of motion for each particle

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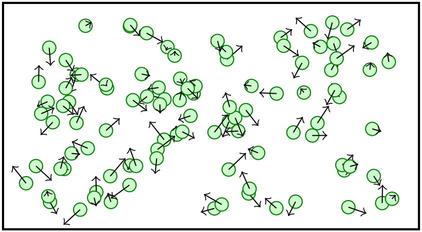

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  - Hardly answers "childish" questions like how hot is it

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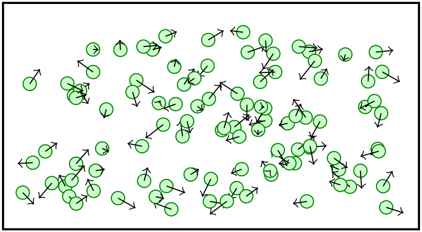

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- Mesoscopic dynamics / Kinetic Theory

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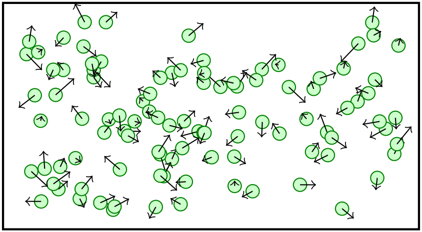

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  Use newtonian equations of motion for each particle

  - Infeasible (recall Avogardo number $N \sim 10^{23}$)
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- Mesoscopic dynamics / Kinetic Theory

  Look at distribution of particles

  - Ask how many particle are doing what
  - Do not ask which particle

Equation of Motion

Physically the dynamic of a system is described by a Hamiltonian

$$H = \underbrace{\sum_{i=1}^{n} \frac{1}{2}\mathbf{v_i}^2}_{\text{kinetic energy}} + \underbrace{\sum_{i=1}^{n} \phi(\mathbf{x}_i)}_{\text{external potential}} + \underbrace{\sum_{i<j} \psi(\mathbf{x}_i - \mathbf{x}_j)}_{\text{Binary interaction}} \tag{1}$$

with equations of motion

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i &= \frac{\partial H}{\partial \mathbf{v}_i} = \mathbf{v}_i \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i &= -\frac{\partial H}{\partial \mathbf{x}_i} = -\nabla\phi(\mathbf{x}_i) - \sum_{i \neq j} \psi(\mathbf{x}_i - \mathbf{x}_j) \end{aligned} \tag{2}$$

Liouville Equation

Consider joint distribution function $F_N(t, \mathbf{x}_1, \mathbf{v}_1, \ldots \mathbf{x}_n, \mathbf{v}_n)$. Claim evolution by

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} \cdot \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} \cdot \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0 \tag{3}$$

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$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} . \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} . \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0 \tag{3}$$

- Expected if considering Poisson brackets
- Constant along trajectories
- Reduces to equation of motion for point masses
- Same amount of information

One Particle Distribution

Try do describe the system by marginal distribution

$$f^{(1)}(t, \mathbf{x}, \mathbf{v}) = \int F_N(t, \mathbf{x}, \mathbf{v}, \mathbf{x}_2, \mathbf{v}_2, \dots \mathbf{x}_n, \mathbf{v}_n) \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \dots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n \qquad (4)$$

and assume symmetry.
Find dynamic equation by integrating Liouville equation

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} . \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} . \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0 \qquad (5)$$

Dynamic Equation for $f^{(1)}$

Recall

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} \cdot \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} \cdot \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0$$

$$H = \sum_{i=1}^{n} \frac{1}{2} \mathbf{v_i}^2 + \sum_{i=1}^{n} \phi(\mathbf{x}_i) + \sum_{i<j} \psi(\mathbf{x}_i - \mathbf{x}_j)$$

Integrate first term

$$\int \frac{\partial F_N}{\partial t}(t, \mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots \mathbf{x}_n, \mathbf{v}_n) \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n = \frac{\partial f^{(1)}}{\partial t}(t, \mathbf{x}_1, \mathbf{v}_1)$$

Dynamic Equation for $f^{(1)}$

Recall

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} . \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} . \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0$$

$$H = \sum_{i=1}^{n} \frac{1}{2} \mathbf{v_i}^2 + \sum_{i=1}^{n} \phi(\mathbf{x}_i) + \sum_{i<j} \psi(\mathbf{x}_i - \mathbf{x}_j)$$

Integrate second term

$$\int \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} . \frac{\partial F_N}{\partial \mathbf{x}_i} \right) \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n = \int \frac{\partial H}{\partial \mathbf{v}_1} . \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

$$= \mathbf{v}_1 . \frac{\partial f^{(1)}}{\partial \mathbf{x}_1}$$

Dynamic Equation for $f^{(1)}$

Recall

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{v}_i} \cdot \frac{\partial F_N}{\partial \mathbf{x}_i} - \frac{\partial H}{\partial \mathbf{x}_i} \cdot \frac{\partial F_N}{\partial \mathbf{v}_i} \right) = 0$$

$$H = \sum_{i=1}^{n} \frac{1}{2} \mathbf{v_i}^2 + \sum_{i=1}^{n} \phi(\mathbf{x}_i) + \sum_{i<j} \psi(\mathbf{x}_i - \mathbf{x}_j)$$

Integrate third term

$$\int \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{x}_i} \cdot \frac{\partial F_N}{\partial \mathbf{v}_i} \right) \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \dots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n = \int -\frac{\partial H}{\partial \mathbf{x}_1} \cdot \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \dots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

$$= -\int \left(\frac{\partial \phi}{\partial \mathbf{x}_1} + \sum_{j=2}^{n} \frac{\partial \psi}{\partial \mathbf{x}_1} (\mathbf{x}_1 - \mathbf{x}_j) \right) \cdot \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \dots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

Dynamic Equation for $f^{(1)}$

Integrate third term

$$\int \sum_{i=1}^{n} \left(\frac{\partial H}{\partial \mathbf{x}_i} . \frac{\partial F_N}{\partial \mathbf{v}_i} \right) \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n = \int -\frac{\partial H}{\partial \mathbf{x}_1} . \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

$$= -\int \left(\frac{\partial \phi}{\partial \mathbf{x}_1} + \sum_{j=2}^{n} \frac{\partial \psi}{\partial \mathbf{x}_1} (\mathbf{x}_1 - \mathbf{x}_j) \right) . \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

$$= -\frac{\partial \phi}{\partial \mathbf{x}_1}(\mathbf{x}_1) . \frac{\partial f^{(1)}}{\partial \mathbf{v}_1} - (n-1) \int \frac{\partial \psi}{\partial \mathbf{x}_1} (\mathbf{x} - \mathbf{x}_2) . \frac{\partial F_N}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

$$= -\frac{\partial \phi}{\partial \mathbf{x}_1}(\mathbf{x}_1) . \frac{\partial f^{(1)}}{\partial \mathbf{v}_1} - (n-1) \int \frac{\partial \psi}{\partial \mathbf{x}_1} (\mathbf{x} - \mathbf{x}_2) . \frac{\partial f^{(2)}}{\partial \mathbf{x}_1} \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{v}_2$$

where

$$f^{(2)}(t, \mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{x}_2) = \int F_N(t, \mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots \mathbf{x}_n, \mathbf{v}_n) \mathrm{d}\mathbf{x}_3 \mathrm{d}\mathbf{v}_3 \ldots \mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{v}_n$$

Dynamic Equation for $f^{(1)}$

Putting everything together gives

$$\begin{aligned}\frac{\partial f^{(1)}}{\partial t}(t,\mathbf{x},\mathbf{v}) + \mathbf{v}.\frac{\partial f^{(1)}}{\partial \mathbf{x}}(t,\mathbf{x},\mathbf{v}) - \frac{\partial \phi}{\partial \mathbf{x}}(x).\frac{\partial f^{(1)}}{\partial \mathbf{v}}(t,\mathbf{x},\mathbf{v}) \\ -(n-1)\int \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{y})\frac{\partial f^{(2)}}{\partial \mathbf{v}}(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u} = 0\end{aligned} \tag{6}$$

- BBGKY hierarchy
- Need to break chain down
- Use Molecular Chaos (Boltzmann) as $n \to \infty$

$$f^{(2)} = f^{(1)} \otimes f^{(1)} \tag{7}$$

- Need appropriate scaling as $n \to \infty$

Mean Field Approximation

Ignore collisions and assume particle only influenced by mean field

$$
\begin{aligned}
(n-1)\int \frac{\partial \psi}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{y})\frac{\partial f^{(2)}}{\partial \mathbf{v}}(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{u} \approx -\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x})\frac{\partial f^{(1)}}{\partial \mathbf{v}} \\
V = -\int \psi(\mathbf{x}-\mathbf{y})f^{(1)}(\mathbf{y},\mathbf{v})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}
\end{aligned}
\tag{8}
$$

With Coulomb interaction $\psi(\mathbf{x}-\mathbf{y}) \propto \frac{1}{|\mathbf{x}-\mathbf{y}|}$ and recalling fundamental solution to Poisson equation

$$
\nabla^2 V(t,\mathbf{x}) = \int f^{(1)}(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v} \tag{9}
$$

Vlasov Equation

Including the background potential we arrived at **Vlasov equation**

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} - \nabla V . \frac{\partial f}{\partial \mathbf{v}} = 0$$
$$\nabla^2 V = \rho_{ions} - \int f \mathrm{d}v \tag{10}$$

Assume overall neutrality, i.e.

$$\int \rho_{ions} \mathrm{d}\mathbf{x} = \int f \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \tag{11}$$

Maxwell Distribution

The equilibrium distribution at temperature T is the *Maxwell distribution* f_0

$$f_0(\mathbf{v}) \propto e^{-\mathbf{v}^2/2T} \tag{12}$$

- Follows from Boltzmann distribution / canonical ensemble
- Maxwell's original argument uses proposed symmetry
- Boltzmann H-Theorem explains why with collisions the system relaxes towards the Maxwell distribution

Linearised Equation around Maxwell Distribution

We want to consider a small perturbation h around equilibrium

$$f(t, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{v}) + h(t, \mathbf{x}, \mathbf{v}) \tag{13}$$

Vlasov equation

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} - \nabla V . \frac{\partial f}{\partial \mathbf{v}} &= 0 \\ \nabla^2 V = \rho_{ions} - \int f \, \mathrm{d}v \end{aligned} \tag{14}$$

Linearised Vlasov equation

$$\begin{aligned} \frac{\partial h}{\partial t} + \mathbf{v}\frac{\partial h}{\partial \mathbf{x}} - \nabla V . \frac{\partial f_0}{\partial \mathbf{v}} &= 0 \\ \nabla^2 V = - \int h \, \mathrm{d}v \end{aligned} \tag{15}$$

Assuming contribution from ions cancels with contribution from f_0.

Split into Fourier Modes

Linearised Vlasov equation

$$\frac{\partial h}{\partial t} + \mathbf{v}\frac{\partial h}{\partial \mathbf{x}} - \nabla V . \frac{\partial f_0}{\partial \mathbf{v}} = 0$$
$$\nabla^2 V = -\int h \, \mathrm{d}v$$

These equations are linear, so consider Fourier modes separately. Consider:

$$h_{\mathbf{k}}(v, t) e^{i(\mathbf{kx})}$$

Mode equation

Dropping index $\mathbf{k}$, and assume k along x-axis:

$$\frac{\partial h}{\partial t} + ikv_x h - ikV \frac{\partial f_0}{\partial v_x} = 0$$
$$k^2 V = \int h \, \mathrm{d}\mathbf{v}$$

Solving using Laplace Transformation

Introduce Laplace transformation in time

$$h_p(\mathbf{v}) = \int_0^\infty h(\mathbf{v}, t) e^{-pt} \mathrm{d}t \tag{16}$$

then ($\sigma > 0$):

$$h(t, \mathbf{v}) = \int_{-i\infty+\sigma}^{i\infty+\sigma} h_p(\mathbf{v}) e^{pt} \mathrm{d}p \tag{17}$$

Solution in Laplace Transformation

Using integration by parts and simple algebra we find with initial condition $g(\mathbf{v}) = h(t, \mathbf{v})$:

$$h_p(\mathbf{v}) = \frac{1}{p + ikv_x} \left(g(\mathbf{v}) + ikV_p \frac{\partial f_0(\mathbf{v})}{\partial v_x} \right)$$

$$V_p = \frac{1}{k^2} \cdot \frac{\int \frac{g(\mathbf{v})}{p+ikv_x} \mathrm{d}\mathbf{v}}{1 - \frac{i}{k^2} \int \frac{\partial f_0}{\partial v_x} \frac{\mathrm{d}\mathbf{v}}{(p+ikv_x)}}$$

Simplify Solution

Integrate out trivial directions $\mathrm{d}v_y, \mathrm{d}v_z$

$$g(u) = \int g(\mathbf{v})\mathrm{d}v_y\mathrm{d}v_z \tag{18}$$

Find

$$V_p = \frac{1}{k^2} \cdot \frac{\int_{-\infty}^{\infty} \frac{g(u)}{p+iku}\mathrm{d}u}{1 - \frac{i}{k^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}f_0}{\mathrm{d}u}\frac{\mathrm{d}u}{(p+iku)}}$$

$$f_0(u) = n\sqrt{\frac{1}{2\pi T}}e^{-\frac{u^2}{2T}}$$

Investigate Solution

Laplace transformation

$$h_p = \int_0^\infty h(t) e^{-pt} \mathrm{d}t$$

only defined for p in the right half plane. Extend through analytic continuation.

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$$V_p = \frac{1}{k^2} \cdot \frac{\int_{-\infty}^{\infty} \frac{g(u)}{p+iku} \mathrm{d}u}{1 - \frac{i}{k^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}f_0}{\mathrm{d}u} \frac{\mathrm{d}u}{(p+iku)}}$$

has poles p_k at

$$\frac{i}{k^2} \int \frac{\mathrm{d}f_0}{\mathrm{d}u} \frac{\mathrm{d}u}{(p+iku)} = 1 \qquad (19)$$

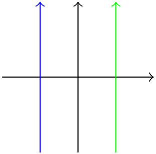
As $\frac{\mathrm{d}f_0}{\mathrm{d}u} > 0$, all poles are in the left half plane.

Asymptotic Solution

Recall inversion formula ($\sigma > 0$)

$$h(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{\sigma} h_p e^{pt} \mathrm{d}p$$

Shift contour into the left half plane



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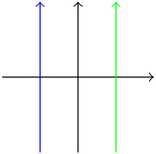


For $t \to \infty$ only contributions from poles:

- poles are in the left half plane
- decaying potential V

Limiting behaviour

So we can summarise the limiting behaviour as $t \to \infty$

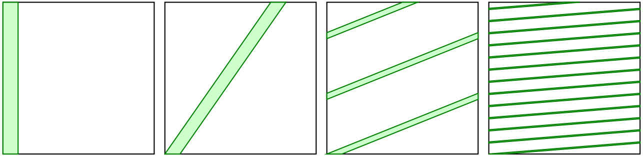
- Potential V is decaying
- Perturbation h is **not** decaying

Time Reversibility

Vlasov equation is time-reversible, so if there are decaying modes there should also be growing modes

- Linearised Equation is not time-reversible
- How does the non-linear term evolve?

Phase Mixing



- Strong vs. Weak topology
  Only the potential V is decaying. The perturbation h is oscillating.

Outlook

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  Penrose Criterion

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- For what equilibrium distribution does the linear damping occur?
  Penrose Criterion
- Recently Mouhot and Villani gave a theorem for non-linearised theory

Acknowledgement

I would like to thank Dr Mouhot for introducing me to this area.