Part III Essay - Penrose Stability Criterion for Plasmas

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Contents

1 Introduction

A plasma is a system that like a gas consists of nearly free moving particles and where additionally some particles are charged. In astrophysics [5, 22] this is a very common state of matter, e.g. what we see from the sun is a plasma. Also in technical applications on earth it is common, e.g. in modern galvanization, in fluorescent lamps, or in attempts to build a fusion reactor. Even the freely moving electrons in a metal can be considered as a plasma.

For some plasmas collisions are in the observed time frame negligible and we can describe the dynamic with the time-reversible Vlasov equation. This Vlasov equation has many solutions constant in space and time and we are interested in how the plasma reacts to small perturbations.

In some cases, most notable in the case of a Maxwell distribution, we find Landau damping, i.e. perturbations seems to decay. This is an exciting phenomena of plasma physics, since it shows how damping of a wave can occur without dissipation, e.g. collisions. Like the theoretical prediction of electromagnetic waves by Maxwell which were only later experimentally found by Hertz, this is an example how admirable mathematical analysis leads to remarkable new results in physics.

A condition when a solution constant in time and space is stable under perturbations following the linearised equation is the Penrose stability criterion. Up to the boundary case it is a sharp criterion, i.e. otherwise there exist growing modes.

This essay aims to give a mathematical and physical self-contained treatment of linear Landau damping. We will roughly follow the historical development and see the collective effort from different people. However, most care is taken of a consistent and logical development. Furthermore, we give a careful mathematical discussion.

After a historical overview in section 2 we will proceed with the physical modelling in section 3 where we will ignore collisions, magnetic fields and use the electrostatic approximation. The notation used is summarised in appendix A which should be sufficient for readers not interested in the physical modelling in section 3. Then we will treat linearised Landau damping in section 4 where we will use general mathematical results which are discussed in section 5. In the physical modelling (section 3) we proceed purely formally. However, the results of section 4 and section 5 are fully proved.

Mathematically, we rewrite the initial value problem into a Volterra equation using the Duhamel principle and then derive stability results. Readers who are interested in the mathematical development could read section 5 first and then appendix A and section 4.

2 Historical Overview

In the late 19$^{\text{th}}$ century Boltzmann and Maxwell developed the kinetic theory of gases. Boltzmann was able to give an explanation with his famous H-Theorem how collisions generate entropy forcing the system towards the thermal equilibrium. However, the idea of atoms was still controversial. Furthermore, the idea of physics at this time was mostly about deterministic laws. For example Planck first rejected Boltzmann's probabilistic interpretation of the second law of thermodynamics and tried to give a deterministic formulation, cf. [9].

Later on, attempts were made to apply these results to electron gases (an example of a plasma). Landau [13] in 1936 used Debye screening to explain collisions with Coulomb's law.

In 1938, Vlasov recognised that collective effects are essential and that there is a physical meaningful regime in which collisions can be neglected. With a naive normal mode analysis in the linearised case he also showed that a plasma can carry waves. A later english reprint is [27].

In 1946, Landau [14] pointed out that this analysis is insufficient and solved the linearised problem using Laplace transformation. For perturbations around the Maxwell distributions he showed his famous Landau damping.

In 1960, Oliver Penrose [20] gave a stability criterion, i.e. for which distribution perturbations following the linearised dynamic are bounded and related this to the marginal distribution. He also gave a different proof for stability where he quoted a general result on the solution of the Volterra equation. In the same year a rigorous treatment of linearised Landau damping was given by Backus [1] who justified the use of Laplace transformation and pointed out the limit of linearisation.

In 1964, one of the earliest experimental verification was published in [30] using ion acoustic waves which consist of ions and electrons moving in phase.

In 2011, Mouhot and Villani [17] proved a theorem for non-linear Landau damping and found a different proof of linear Landau damping which also makes a more quantitative statement about the decay.

3 Physical Model of a Plasma

This section develops the physical motivation to study the Vlasov equation. We therefore proceed formally. Also we will use the notation $\frac{\partial f(x,y)}{\partial x}$ for a partial derivative with respect to the first variable at the point (x, y).

A brief summary of the required notation for the later sections is given in appendix A.

3.1 Introduction to Collision Free Plasmas

A plasma is a state of matter with freely moving particles of which some are charged (ionised). As an example we can consider (nearly) free electrons in a metal. Another possibility to create a plasma is to heat up a gas sufficiently so that the thermal energy is high enough to ionise a portion of the gas atoms. We can also supply the ionisation energy by a high voltage applied to the gas as used for fluorescent lamps [12].

Having charged particles, the plasma particle interact through the strong electromagnetic force (e.g. between two electrons the Coulomb force is 10^{42} times stronger than gravity, between two single ionised Nitrogen molecules N_2 it is 10^{34} times stronger). Also the plasma is subject to magnetic fields yielding to beautiful effects like aurora borealis or solar flares.

In this essay we will restrict ourself to collision free plasmas with the electrostatic approximation for the interaction between the particles and only develop the kinetic theory necessary to discuss Landau damping. A theoretical overview of plasma physics may be found in [23]. Introductions more focused on astrophysical applications are [5, 22].

3.2 Finding Collective Effects – Levels of Description

Today the most fundamental description of a physical system is believed to be quantum mechanics. However, the de Broglie wavelength is so small compared to the average distance between two particles in a plasma that we expect to be able to describe the dynamics of the particles with Newtonian mechanics. Combined with Maxwell's equations and a model about collisions (e.g. hard spheres) we can in principle formulate the equations of motion and try to solve them.

Recalling Avogadro's number ($\approx 10^{21}$) for the typical number of atoms in macroscopic systems, we see that this in infeasible. Also we can only observe collective phenomena and not single particle of a plasma. Thus we change the level of description for collective effects. An often encountered model is the continuum model used in fluid mechanics where we describe the system with smooth fields of velocity, density, etc. and use the Navier-Stokes equation (or some modification) to predict the dynamics.

In between we will develop the kinetic theory or mesoscopic dynamics. Here we will still think about particles, but not ask for their individual dynamic. Rather we will ask how many particles are in some position and velocity range.

3.3 Newtonian Description as Starting Point

In order to derive the kinetic theory we start with Newtonian mechanics. Let $\mathbf{p}$ and $\mathbf{q}$ be the canonical conjugate position and momentum, i.e. in our case without magnetic field $\mathbf{p} = (p_1, p_2, \dots p_N)$ and $\mathbf{q} = (q_1, q_2, \dots q_N)$ with $q_i = m_i v_i$ where p_i is the position (in cartesian coordinates), v_i the velocity, and m_i the mass of the i^{th} particle. The phase space Γ is the joint vector space of position and momentum so that a point $(\mathbf{p}, \mathbf{q})$ fully describes the system and we write $\mathbf{X} = (\mathbf{p}, \mathbf{q})$. Then in Hamilton's formulation the dynamic of the system $\mathbf{X} = (\mathbf{p}, \mathbf{q})$ is given by

$$\dot{\mathbf{p}} := \frac{\partial \mathbf{p}}{\partial t} = \frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial \mathbf{q}}, \qquad \dot{\mathbf{q}} := \frac{\partial \mathbf{q}}{\partial t} = -\frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial \mathbf{p}} \tag{3.1}$$

where $H(\mathbf{p}, \mathbf{q})$ is the Hamiltonian which is a constant of motion. The value of the Hamiltonian is interpreted as the energy of the system.

In our case

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{m_i} + \sum_{i=1}^{N} e_i \psi(q_i) + \sum_{i=1}^{N} \sum_{j=1}^{i-1} \psi_{ij}(q_i, q_j) \tag{3.2}$$

where e_i is the charge of the i^{th} particle, ψ is the external electric potential and ψ_{ij} is the interaction energy which is symmetric (i.e. $\psi_{ij}(q_i, q_j) = \psi_{ji}(q_j, q_i)$) and given in the electrostatic approximation by Coulomb's law

$$\psi_{ij}(q_i, q_j) = \frac{e_i e_j}{|q_i - q_j|} \tag{3.3}$$

where we choose the unit of charge such that $4\pi\epsilon = 1$. The first sum $(1/2) \sum_{i=1}^{N} p_i^2/m_i$ is the kinetic energy while the other terms are the potential energy.

3.4 Evolution of a Distribution of States – Liouville Equation

We now suppose we have a distribution $F_N(t, \mathbf{p}, \mathbf{q})$ of states. In statistical physics we think of an ensemble of systems where each system evolves independently in time and $F_N(t, \mathbf{p}, \mathbf{q})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$ describes the proportion of systems with position in $[\mathbf{p}, \mathbf{p} + \mathrm{d}\mathbf{p}]$ and momentum in $[\mathbf{q}, \mathbf{q} + \mathrm{d}\mathbf{q}]$ at time t.

Assuming that each system evolves continuously, we impose for each volume V in the phase space Γ

$$\frac{\partial}{\partial t} \int_V F_N(t, \mathbf{X}) \mathrm{d}V + \oint_{\delta V} F_N(t, \mathbf{X}) \dot{\mathbf{X}} \cdot \mathrm{d}\mathbf{s} = 0 \tag{3.4}$$

where $\mathbf{X} = (\mathbf{p}, \mathbf{q})$, $\dot{\mathbf{X}} = \frac{\partial \mathbf{X}}{\partial t}$, and $\mathrm{d}\mathbf{s}$ is the surface element pointing outwards. Using Gauss law we find

$$\frac{\partial}{\partial t} \int_V F_N(t, \mathbf{X}) \mathrm{d}V + \int_V \nabla \cdot (F_N(t, \mathbf{X}) \dot{\mathbf{X}}) \mathrm{d}V = 0. \tag{3.5}$$

Since this holds for all V we conclude the continuity equation

$$\frac{\partial F_N(t, \mathbf{X})}{\partial t} + \nabla \cdot (F_N(t, \mathbf{X}) \dot{\mathbf{X}}) = 0. \tag{3.6}$$

By Hamilton's equations of motion

$$\nabla \cdot \dot{\mathbf{X}} = \sum_{i=1}^{N} \left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} \right) = \sum_{i=1}^{N} \left(-\frac{\partial}{\partial p_i} \frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial q_i} + \frac{\partial}{\partial q_i} \frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial p_i} \right) = 0 \tag{3.7}$$

which corresponds to an incompressible fluid in phase space.

Putting it together we find Liouville's equation

$$\begin{aligned} 0 &= \frac{\partial F_N(t, \mathbf{X})}{\partial t} + \sum_{i=1}^{N} \left(\dot{p}_i \frac{\partial F_N(t, \mathbf{X})}{\partial p_i} + \dot{q}_i \frac{\partial F_N(t, \mathbf{X})}{\partial q_i} \right) \\ &= \frac{\partial F_N(t, \mathbf{X})}{\partial t} + \sum_{i=1}^{N} \left(-\frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial q_i} \frac{\partial F_N(t, \mathbf{X})}{\partial p_i} + \frac{\partial H(\mathbf{p}, \mathbf{q})}{\partial p_i} \frac{\partial F_N(t, \mathbf{X})}{\partial q_i} \right) \end{aligned} \tag{3.8}$$

where the sum is also denoted with the Poisson bracket $\{F_N, H\}$ so that it takes the form expected from classical mechanics for a conserved quantity.

Another approach comes from Liouville's theorem. Let $S(t_2,t_1)$ be the time evolution operator from time t_1 to t_2, i.e. $\mathbf{X}(t) = S(t,t_1)\mathbf{X}$ is a solution of the equations of motion and $\mathbf{X}(t_1) = \mathbf{X}$. Then look at the Jacobian $J(t)$ of $S(t,t_1)$. Clearly $J(t_1) = 1$ and with $\mathbf{X}(t) = (\mathbf{p}(t),\mathbf{q}(t))$

$$\begin{aligned}
\left.\frac{\partial J}{\partial t}\right|_{t_1} &= \frac{\partial}{\partial t}\det\left.\begin{pmatrix} \frac{\partial \mathbf{p}(t)}{\partial \mathbf{p}} & \frac{\partial \mathbf{p}(t)}{\partial \mathbf{q}} \\ \frac{\partial \mathbf{q}(t)}{\partial \mathbf{p}} & \frac{\partial \mathbf{q}(t)}{\partial \mathbf{q}} \end{pmatrix}\right|_{t_1} \\
&= \frac{\partial}{\partial \mathbf{p}}\left(\frac{\partial \mathbf{p}}{\partial t}\right)_{t_1} + \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathbf{q}}{\partial t}\right)_{t_1} \\
&= \frac{\partial^2 H(\mathbf{p},\mathbf{q})}{\partial \mathbf{p}\partial \mathbf{q}} - \frac{\partial^2 H(\mathbf{p},\mathbf{q})}{\partial \mathbf{q}\partial \mathbf{p}} = 0
\end{aligned} \tag{3.9}$$

where we used that the matrix $\begin{pmatrix} \frac{\partial \mathbf{p}(t)}{\partial \mathbf{p}} & \frac{\partial \mathbf{p}(t)}{\partial \mathbf{q}} \\ \frac{\partial \mathbf{q}(t)}{\partial \mathbf{p}} & \frac{\partial \mathbf{q}(t)}{\partial \mathbf{q}} \end{pmatrix}$ is the identity matrix at t_1. As the evolution operator combines as $S(t_2,t)S(t,t_1) = S(t_2,t_1)$ this result extends to $\frac{\partial J}{\partial t} = 0$ for all t. Hence $J(t) = 1$ for all t, which is Liouville's theorem. Therefore, we expect F_N to be constant along a solution of the equations of motion which is exactly the content of Liouville's equation using the method of characteristics.

3.5 Evolution of Marginal Distributions – BBGKY Hierarchy

The Liouville equation still contains the same complexity and amount of information as a Newtonian description. In order to reduce the complexity we suppose that we have N identical particles each with mass m and charge e and that the particles are indistinguishable, i.e. their distributions are symmetric. This means that for any function $g : \mathbb{R}^{3N} \to \mathbb{R}$ and any permutation ϵ of $\{1,2,\ldots,N\}$ holds

$$\begin{aligned}
&\int g(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{v}_1\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N \\
&\qquad = \int g(t,\mathbf{x}_{\epsilon(1)},\mathbf{v}_{\epsilon(1)},\ldots,\mathbf{x}_{\epsilon(N)},\mathbf{v}_{\epsilon(N)})\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{v}_1\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N.
\end{aligned} \tag{3.10}$$

and for $i \neq j$ holds

$$\psi_{12} = \psi_{ij}. \tag{3.11}$$

Considering the distribution $F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)$ with respect to the position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ of the i^{th} particle, we reduce for $n \leq N$ to the n particle distribution or marginal distribution.

$$f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n) = \int F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N. \tag{3.12}$$

By integrating Liouville's equation over $\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N$ we find the dynamic equation for $f^{(n)}$. We still proceed formally and impose that the physical solutions are sufficiently regular. In our case with the Hamiltonian as in eq. (3.2) we find for the first term

$$\int \frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial t}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N = \frac{\partial}{\partial t}f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n). \tag{3.13}$$

For the second term we find

$$\begin{aligned}
&\int \sum_{i=1}^{N}\left(-\frac{\partial H(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N)}{\partial \mathbf{x}_i}\cdot\frac{1}{m}\frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial \mathbf{v}_i}\right)\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N \\
&= -\sum_{i=1}^{N}\int\left(\frac{e}{m}\frac{\partial \psi(\mathbf{x}_i)}{\partial \mathbf{x}_i} + \sum_{j\neq i}\frac{1}{m}\frac{\partial \psi_{ij}(\mathbf{x}_i,\mathbf{x}_j)}{\partial \mathbf{x}_i}\right)\cdot\frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial \mathbf{v}_i}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N.
\end{aligned} \tag{3.14}$$

For $i \geq n+1$ the integral $\int \frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial \mathbf{v}_i} \mathrm{d}\mathbf{v}_i$ vanishes since $F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)$ is a probability distribution. Hence the sum reduces to $i = 1,\ldots,n$. Let

$$H^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{v}_1,\ldots,\mathbf{v}_n) = \frac{m}{2}\sum_{i=1}^{n}\mathbf{v}_i^2 + \sum_{i=1}^{n} e\psi(\mathbf{x}_i) + \sum_{i=1}^{n}\sum_{j=1}^{i-1}\psi_{ij}(\mathbf{x}_i,\mathbf{x}_j), \tag{3.15}$$

then we can write the term as

$$\begin{aligned}
&\sum_{i=1}^{n}\left(-\frac{\partial H^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{v}_1,\ldots,\mathbf{v}_n)}{\partial \mathbf{x}_i}\cdot\frac{1}{m}\frac{\partial f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)}{\partial \mathbf{v}_i}\right)\\
&\quad -\int\sum_{i=1}^{n}\sum_{j=n+1}^{N}\frac{1}{m}\frac{\partial \psi_{ij}(\mathbf{x}_i,\mathbf{x}_j)}{\partial \mathbf{x}_i}\cdot\frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial \mathbf{v}_i}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N.
\end{aligned} \tag{3.16}$$

By the symmetry of the particles the remaining integral is

$$(N-n)\sum_{i=1}^{n}\int\frac{1}{m}\frac{\partial \psi_{i,n+1}(\mathbf{x}_i,\mathbf{x}_{n+1})}{\partial \mathbf{x}_i}\cdot\frac{\partial f^{(n+1)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_{n+1},\mathbf{v}_{n+1})}{\partial \mathbf{v}_{n+1}}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}. \tag{3.17}$$

Finally the last term is

$$\begin{aligned}
&\int\sum_{i=1}^{N}\left(\frac{1}{m}\frac{\partial H(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N)}{\partial \mathbf{v}_i}\cdot\frac{\partial F_N(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_N,\mathbf{v}_N)}{\partial \mathbf{x}_i}\right)\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N\\
&\quad=\int\sum_{i=1}^{N}\mathbf{v}_i\frac{\partial F_N}{\partial \mathbf{x}_i}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N\\
&\quad=\sum_{i=1}^{n}\mathbf{v}_i\frac{\partial f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)}{\partial \mathbf{x}_i}\\
&\quad=\sum_{i=1}^{n}\frac{1}{m}\frac{\partial H^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{v}_1,\ldots,\mathbf{v}_n)}{\partial \mathbf{v}_i}\cdot\frac{\partial f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)}{\partial \mathbf{x}_i}.
\end{aligned} \tag{3.18}$$

Collecting the terms we arrive at the BBGKY hierarchy [21, 23] (named after Bogoliubov, Born, Green, Kirkwood, and Yvon)

$$\begin{aligned}
&\frac{\partial}{\partial t}f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)+\sum_{i=1}^{n}\left(-\frac{\partial H^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{v}_1,\ldots,\mathbf{v}_n)}{\partial \mathbf{x}_i}\cdot\frac{1}{m}\frac{\partial f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)}{\partial \mathbf{v}_i}\right.\\
&\qquad\left.+\frac{1}{m}\frac{\partial H^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{v}_1,\ldots,\mathbf{v}_n)}{\partial \mathbf{v}_i}\cdot\frac{\partial f^{(n)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_n,\mathbf{v}_n)}{\partial \mathbf{x}_i}\right)\\
&=(N-n)\sum_{i=1}^{n}\int\frac{1}{m}\frac{\partial \psi_{i,n+1}(\mathbf{x}_i,\mathbf{x}_{n+1})}{\partial \mathbf{x}_i}\cdot\frac{\partial f^{(n+1)}(t,\mathbf{x}_1,\mathbf{v}_1,\ldots,\mathbf{x}_{n+1},\mathbf{v}_{n+1})}{\partial \mathbf{v}_i}\mathrm{d}\mathbf{x}_{n+1}\mathrm{d}\mathbf{v}_{n+1}.
\end{aligned} \tag{3.19}$$

Hence the n particle distribution dynamic depends on the $n+1$ particle distribution dynamic giving a hierarchy. Therefore, to find a closed form of the dynamic for the one particle distribution we need to make some further assumption.

3.6 Evolution of the One Particle Distribution – Vlasov Equation

In case of the one particle distribution the BBGKY hierarchy simplifies (relabelling the dummy integration variables $\mathbf{x}_2$ and $\mathbf{v}_2$ as $\mathbf{y}$ and $\mathbf{w}$) to

$$\begin{aligned}\frac{\partial f^{(1)}(t,\mathbf{x},\mathbf{v})}{\partial t} &- \frac{e}{m}\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}\cdot\frac{\partial f^{(1)}(t,\mathbf{x},\mathbf{v})}{\partial\mathbf{v}} + \mathbf{v}\cdot\frac{\partial f^{(1)}(t,\mathbf{x},\mathbf{v})}{\partial\mathbf{x}}\\ &= (N-1)\int\frac{1}{m}\frac{\partial\psi_{12}(\mathbf{x},\mathbf{y})}{\partial\mathbf{x}}\cdot\frac{\partial f^{(2)}(t,\mathbf{x},\mathbf{v},\mathbf{y},\mathbf{w})}{\partial\mathbf{v}}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}.\end{aligned}\tag{3.20}$$

Already in 1872 Ludwig Boltzmann [4, Equation 44] found this equation for gas particles where the RHS is replaced by a collision term which he justified more heuristically. From the structure of the BBGKY hierarchy we see that the key ingredient for a closed form of $f^{(1)}$ is to make some assumption about $f^{(2)}$ in terms of $f^{(1)}$. The simplest assumption would be $f^{(2)} = f^{(1)}\otimes f^{(1)}$ which yields for the RHS with Coulomb interaction $\psi_{12}(\mathbf{x},\mathbf{y}) = e^2\left|\mathbf{x}-\mathbf{y}\right|^{-1}$

$$\begin{aligned}(N-1)&\int\frac{1}{m}\frac{\partial}{\partial\mathbf{x}}\left(\frac{e^2}{|\mathbf{x}-\mathbf{y}|}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\left(f^{(1)}(t,\mathbf{x},\mathbf{v})f^{(1)}(t,\mathbf{y},\mathbf{w})\right)\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}\\ &= \frac{e}{m}\frac{\partial f^{(1)}(t,\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\cdot(N-1)\frac{\partial}{\partial\mathbf{x}}\left(\int\frac{e}{|\mathbf{x}-\mathbf{y}|}f^{(1)}(t,\mathbf{y},\mathbf{w})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}\right)\\ &= \frac{e}{m}\frac{\partial f^{(1)}(t,\mathbf{x},\mathbf{v})}{\partial\mathbf{v}}\cdot\frac{\partial\phi_i(t,\mathbf{x})}{\partial\mathbf{x}}\end{aligned}\tag{3.21}$$

where $\phi_i(t,\mathbf{x}) = (N-1)\int\frac{e}{|\mathbf{x}-\mathbf{y}|}f^{(1)}(t,\mathbf{y},\mathbf{w})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}$ which we recognise from electrostatics as the mean electric potential created by the other particles. It is equivalently described by Gauss's law

$$\nabla^2\phi_i(t,\mathbf{x}) = -4\pi e(N-1)\int f^{(1)}(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}\tag{3.22}$$

which is just the fact that the Laplace equation has the fundamental solution $-(4\pi r)^{-1}$. Hence we find the mean field limit where one particle only notices the average of the other particles.

Strictly speaking $f^{(2)} = f^{(1)}\otimes f^{(1)}$ cannot hold since when two particles approach each other the Coulomb force repels them and at some point they collide. Note that however, for a plasma the number of particles is very large and even localised measurements involve many particles.

Hence we consider the limit $N\to\infty$ which is called the thermodynamic limit. This limit can be thought of as changing the length scale so that a plasma confined in a unit volume becomes a plasma with an increasing number of particles. Mathematically this is the same limit as breaking down each particle into several sub-particles. Hence Ne and Nm should remain constant which implies that $m^{-1}\psi_{12}(\mathbf{x},\mathbf{y})$ scales as N^{-1} balancing the (N-1) in the RHS of the BBGKY hierarchy for one particle in eq. (3.20).

The marginal distributions also depend on N and we explicitly write $f^{(n),N}$ for the n particle distribution of a N particle system. In the thermodynamic limit $N\to\infty$ we now consider

$$f^{(n),\infty} = \lim_{N\to\infty} f^{(n),N}.\tag{3.23}$$

In this limit Boltzmann postulated molecular chaos

$$f^{(2),\infty} = f^{(1),\infty}\otimes f^{(1),\infty}\tag{3.24}$$

and considered for large N collisions to derive his dynamical equation which drives the system to the thermal equilibrium and introduces a direction of time (H-Theorem).

Vlasov [27] however noted that for some systems collisions are so rare for the observed time frame that they can be neglected. Physically, we suppose that the number of particles N is so large that this limit already holds

$$f^{(2),N} \sim f^{(1),N}\otimes f^{(1),N}.\tag{3.25}$$

This is called the mean field limit, because every particle moves as it would only experiences the average of the other particles.

Let $f(t, \mathbf{x}, \mathbf{v})$ be the density of particles in phase space, i.e. $f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$ is the number of particles with position in $[\mathbf{x}, \mathbf{x} + \mathrm{d}\mathbf{x}]$ and velocity in $[\mathbf{v}, \mathbf{v} + \mathrm{d}\mathbf{v}]$. This relates to the previously considered case with finitely many particles as $f(t, \mathbf{x}, \mathbf{v}) = N f^{(1),N}(t, \mathbf{x}, \mathbf{v})$. We consider a distribution with $\int f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} = \infty$ as limit case of a large confining box, so that we can have a spatial homogeneous distribution. Otherwise, we may introduce a length scale L and apply periodic boundary conditions we take the space to be a torus $\mathbb{T}_L^d = \mathbb{R}^d / L\mathbb{Z}^d$ of side length L, i.e. $[0, L]^d$ with opposite sites identified.

Finally, we assume that the external potential ψ is created by a background charge distribution ρ_b and independent of time[1],i.e.

$$\nabla^2 \psi(\mathbf{x}) = -4\pi \rho_b(\mathbf{x}). \tag{3.26}$$

Then the mean field limit is the Vlasov equation

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m} \frac{\partial \phi(t, \mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} = 0 \tag{3.27}$$

$$\nabla^2 \phi(t, \mathbf{x}) = -4\pi \left(\rho_b + e \int f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v} \right) \tag{3.28}$$

or with the electric field $\mathbf{E} = -\nabla\phi$ we can replace the second equation with

$$\nabla \cdot \mathbf{E}(t, \mathbf{x}) = 4\pi \left(\rho_b + e \int f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v} \right). \tag{3.29}$$

This mean field is the electric field we would measure if we take a macroscopic measurement of the electric field.

If we would have different kind of particles, we use greek indices. So let f_α, m_α, e_α respectively the phase space density, mass, and charge for the particles of kind α. By the same argument [23, Chapter 4] we get

$$\frac{\partial f_\alpha(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} - \frac{e_\alpha}{m_\alpha} \frac{\partial \phi(t, \mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f_\alpha(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} = 0 \tag{3.30}$$

$$\nabla^2 \phi(t, \mathbf{x}) = -4\pi \left(\rho_b + \sum_\alpha e_\alpha \int f_\alpha(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v} \right). \tag{3.31}$$

Since we only consider electric interaction, we can ignore neutral particles in this discussion. In the following analysis the results of one kind of particle easily extends to the case of several kinds of particles. For better readability and clearer discussion we will therefore do most discussions for one kind of (ionised) particle.

From statistical physics an important length scale is the Debye length

$$\lambda_D = \sqrt{\frac{k_B T}{4\pi n e^2}}. \tag{3.32}$$

where $n(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v}$ is the space density of particles, T is the temperature and k_B is the Boltzmann constant. The main thermal velocity is

$$v_t = \sqrt{\frac{k_B T}{m}} \tag{3.33}$$

which allows the definition of the plasma frequency

$$\omega_p = \frac{v_t}{\lambda_D} = \sqrt{\frac{m}{4\pi n e^2}}. \tag{3.34}$$

[1]This ensures that the Hamiltonian of the system is time independent. Hence by Noether's theorem the total energy is conserved.

Vlasov [27] estimated the effect of collisions using the earlier work of Landau [13] showing that for the ionosphere (the system he considered) collisions are negligible. A brief summary is given in [23, Chapter 1] which shows that the ratio of collective effects to individual effects is roughly proportional to the plasma parameter $\Lambda = \frac{4\pi}{3} n \lambda_D^3$. Hence we consider $\Lambda \gg 1$ or the limit $\Lambda \to \infty$.

Another approach is the averaging of the Klimontovich equations. This is extensively used in [22] and clearly described in [23, Chapter 4], whose discussion we will repeat here.

We start by considering the exact distribution function F or empirical measure for the system with N particles given by

$$F(t, \mathbf{x}, \mathbf{v}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)). \tag{3.35}$$

where $\mathbf{x}_i$ and $\mathbf{v}_i$ are the position respectively the velocity of the i^{th} particle.

The particles follow the exact equation of motion

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_i(t)}{\mathrm{d}t} &= \mathbf{v}_i(t) \\ \frac{\mathrm{d}\mathbf{v}_i(t)}{\mathrm{d}t} &= \frac{e}{m}\mathbf{E}(t, \mathbf{x}_i) = -\frac{1}{m} \sum_{j \neq i} \nabla \psi_{ij}(\mathbf{x}_i, \mathbf{x}_j) \end{aligned} \tag{3.36}$$

where $\mathbf{E}(t, \mathbf{x}_i)$ is the electric field.

We can calculate

$$\begin{aligned} &\frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} + \frac{e}{m}\mathbf{E}(t, \mathbf{x}) \cdot \frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} \\ &= \sum_{i=1}^{N} \left[\left(\mathbf{v} - \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} \right) \delta'(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)) + \left(\frac{e}{m}\mathbf{E}(t, \mathbf{x}) - \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} \right) \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta'(\mathbf{v} - \mathbf{v}_i(t)) \right]. \end{aligned} \tag{3.37}$$

Assuming that every particle has a different position in phase space, this is vanishing if and only if the equations of motions are satisfied. Hence equivalently we can impose

$$\frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} + \frac{e}{m}\mathbf{E}(t, \mathbf{x}) \cdot \frac{\partial F(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} = 0. \tag{3.38}$$

Now suppose we have some averaging procedure, e.g. considering an ensemble of systems, which we can mathematically express as measure in the phase space. Using this average we can split F into a mean distribution $f = \langle F \rangle$ and a deviation δF

$$F = f + \delta F \tag{3.39}$$

where we impose $\langle \delta F \rangle = 0$. From the superposition principle we can accordingly split the electric field

$$\mathbf{E} = \langle \mathbf{E} \rangle + \delta \mathbf{E}. \tag{3.40}$$

Averaging the dynamic equation we find

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{x}} + \frac{e}{m} \langle \mathbf{E} \rangle \cdot \frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} = -\frac{e}{m} \left\langle \delta \mathbf{E} \cdot \frac{\partial \delta F(t, \mathbf{x}, \mathbf{v})}{\partial \mathbf{v}} \right\rangle \tag{3.41}$$

where the LHS is the Vlasov equation and the RHS is the neglected collision term. For an estimation of the collision term, consider the gedankenexperiment("thought experiment") of breaking each particle into sub-particles and the limit $N \to \infty$ with

$$Ne = \text{const}, \qquad Nm = \text{const}, \qquad v_t = \text{const}. \tag{3.42}$$

Then e/m remains constant, the temperature T grows as N, and the Debye length λ_D stays constant.

From general statistics we expect $\delta F \sim N^{1/2}$ and by the electrostatic solution

$$\delta\mathbf{E}(t,\mathbf{x}) = \int \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, e\, \delta F(t,\mathbf{y})\mathrm{d}\mathbf{y} \tag{3.43}$$

also $\delta\mathbf{E} \sim N^{-1/2}$. Hence the RHS stays at the same order of magnitude. However, the LHS grows as N. Therefore, in the limit $N \to \infty$ which corresponds to $\Lambda \sim N \to \infty$ the RHS is negligible.

3.7 Statistical Mechanics for Comparison

In statistical mechanics and thermodynamics we consider systems in equilibrium and the evolution is towards this equilibrium which defines a direction of time. Boltzmann's H-Theorem shows that collisions are the driving mechanism for this time irreversible evolution towards equilibrium. In the Vlasov equation however, we are interested in a time scale where collisions are negligible and we have a time reversible equation. Therefore, we cannot apply thermodynamic arguments and the entropy is constant [3, Appendix A].

Assuming some collisional effects we would find the system in the thermal equilibrium at the spatial homogeneous Maxwell distribution

$$f(\mathbf{v}) = n\sqrt{\frac{m}{2\pi k_B T}}e^{-\frac{m\mathbf{v}^2}{2k_B T}}. \tag{3.44}$$

At temperature T the mean velocity is

$$v_t = \sqrt{\frac{k_B T}{m}}. \tag{3.45}$$

Another effect is the Debye screening. Intuitively charged particles repels equally charged particles and attracts oppositely charged particles compensating its own field. Precisely formulated as Debye screening, the effective potential at distance r is $(e/r)e^{-r/\lambda_D}$ where $\lambda_D = \sqrt{k_B T/(4\pi n e^2)}$ is the Debye length.

3.8 Electron Gas as Common Example

A common theoretical example for a collision free plasma is the electron gas where we have freely moving electrons and ions as in a simple model for metals. Since the metal is overall neutral we have in average the same density n of ions and electrons. Due to their much heavier mass the ions can be assumed stationary for high frequency movements and we can approximate the effect of the ions as constant background potential, leaving us with the dynamical equation of the electron distribution.

4 Linearised Landau Damping

With this model of a collision free plasma the evolution becomes a mathematical problem described by the Vlasov equation, which we want to solve. In this essay we ask what happens to a constant solution of the Vlasov equation if we perturbe it. In order to solve this problem we linearise the Vlasov equation around the constant solution which was already done in Vlasov's first paper from 1938. Landau formally solved the problem and found the remarkable result that around the Maxwell distribution perturbations are damped which we now call Landau damping. In this section we will prove this result and develop the Penrose criterion when perturbation are not growing. For this we need results from mathematical analysis which will be developed in section 5.

Without external fields and boundary conditions, any spatial homogenous density $f_0(\mathbf{v})$ with cancelling background charge $\rho_b = \int f(t,\mathbf{v})\mathrm{d}\mathbf{v}$ is a time-independent solution of the Vlasov equation because then $\phi = 0$ and $\frac{\partial f_0}{\partial \mathbf{x}} = 0$. This again shows the difference to the kinetic equation with collisions, where we have an increasing quantity, the entropy, with unique maximum.

4.1 Linearisation

We are now asking what happens if we have a small perturbation $f_1(t,\mathbf{x},\mathbf{v})$ around a constant density[2] $f_0(\mathbf{v})$, i.e. we suppose the density $f(t,\mathbf{x},\mathbf{v}) = f_0(\mathbf{v}) + f_1(t,\mathbf{x},\mathbf{v})$ satisfies Vlasov's equation and look at the evolution of $f_1(t,\mathbf{x},\mathbf{v})$. Assuming neutrality $\rho_b = -e\int f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}$ we find

$$\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi}{\partial \mathbf{x}}\left(\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} + \frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{v}}\right) = 0, \tag{4.1}$$

$$\nabla^2\phi = -4\pi e\int f_1(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}. \tag{4.2}$$

For a small perturbation $f_1 \ll f_0$ and $\frac{\partial f_1}{\partial \mathbf{v}} \ll \frac{\partial f_0}{\partial \mathbf{v}}$ we then look at the linearised Vlasov equation

$$\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}}\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{4.3}$$

$$\nabla^2\phi(t,\mathbf{x}) = -4\pi e\int f_1(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}. \tag{4.4}$$

Similar for several kind of particles we find

$$\frac{\partial f_{1\alpha}(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{1\alpha}(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}}\cdot\frac{\partial f_{0\alpha}(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{4.5}$$

$$\nabla^2\phi(t,\mathbf{x}) = \sum_\alpha -4\pi e_\alpha\int f_{1\alpha}(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}. \tag{4.6}$$

As Backus [1] noted there is no dissipative term, so that we cannot justify a priori a linearised treatment and we can observe non-linear effects like filamentation. This question remained open until in 2011 Mouhot and Villani could answer the non-linear case.

4.2 Naive Normal Modes

By separation of variables Vlasov [27] motivated the ansatz $f_1(t,\mathbf{x},\mathbf{v}) = c(\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$. From the fundamental solution for ϕ, we see that ϕ is proportional to $e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$. The linearised equations become

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})f_1(t,\mathbf{x},\mathbf{v}) - \frac{e}{m}\frac{\partial \phi}{\partial \mathbf{x}}\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} = 0 \tag{4.7}$$

and

$$-\mathbf{k}^2\phi = -4\pi e\int f_1(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}. \tag{4.8}$$

Hence formally we have a solution if

$$\mathbf{k}^2 = \frac{4\pi e^2}{m}\int\frac{\mathbf{k}\cdot\frac{\partial f_0(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{v}}}{\mathbf{k}\cdot\mathbf{v}-\omega}\mathrm{d}\mathbf{v}. \tag{4.9}$$

Choosing the x-axis along $\mathbf{k}$, we can absorb the constant into the (unperturbed) plasma frequency $\omega_p = \sqrt{4\pi ne^2/m}$ with the (unperturbed) density $n = \int f_0(\mathbf{v})\mathrm{d}\mathbf{v}$ and rewrite the condition as

$$k^2 = \int_{-\infty}^{\infty}\frac{\omega_p^2 g_0(u)}{u-\omega/k}\mathrm{d}u \tag{4.10}$$

[2]This should not be confused with the initial distribution of the perturbation which we will denote f_{in}.

where $k = |\mathbf{k}|$ and

$$g_0(u) = \frac{1}{n} \int f_0(u, v_x, v_y) \mathrm{d}v_x \mathrm{d}v_y. \tag{4.11}$$

The mode $k = 0$ corresponds to a shift of density violating overall neutrality, so that we assume $k \neq 0$. Also the perturbation is everywhere small only if k is real.

If ω is real, the denominator $u - \omega/k$ has a pole. However, for complex ω with $\Re(\omega) \neq 0$ we have a valid solution if the condition holds and we can have a perturbation with this frequency.

Beside the potential pole, the more serious problem is that this is not a proper dispersion relation. As in [23, Chapter 4] consider for example

$$f_0(\mathbf{v}) = \delta(v_x - u_1) + \delta(v_x - u_2) + \delta(v_x - u_3) + \delta(v_x - u_4) \tag{4.12}$$

then

$$\int_{-\infty}^{\infty} \frac{\omega_p^2 g_0(u)}{u - \omega/k} \mathrm{d}u = \omega_p^2 \left(\frac{1}{u_1 - \omega/k} + \frac{1}{u_2 - \omega/k} + \frac{1}{u_3 - \omega/k} + \frac{1}{u_4 - \omega/k} \right) \tag{4.13}$$

thus considered as function of ω this has four poles. Hence for all k, there exist at least four possible frequencies. This shows that there is more than one frequency in contrast to a proper dispersion relation.

This contrast comes from the fact that we found only an implicit non-linear integral equation for ω and is mathematically not surprising. However, this differs from common normal mode analysis in physics. This analysis by Vlasov in 1938 already shows that the spatial Fourier modes decouple which is the key ingredient to all further progress.

4.3 Landau's Solution

In 1946, Landau [14] pointed out the insufficiency of the previous analysis by Vlasov whose main problems are that it is unclear what the divergent integrals mean, whether all perturbations have this form, and which frequency to choose. Landau noted that this relates to the fact that we actually want to know the evolution of an initial distribution and therefore should consider it as an initial value problem or Cauchy problem. He proceeded by formally solving this initial value problem and then applied his solution to perturbations around the Maxwell distribution where he showed his Landau damping. We will repeat his formal discussion to motivate Penrose's argumentation. Furthermore, Backus [1] showed that it is possible to do this discussion rigorously without principle trouble as discussed in section 4.6.

We assume that f_1 and ϕ have a Fourier transformation in space. In physics this is almost always assumed and, since a sufficient condition already is integrability, this is physically reasonable.

The linearised Vlasov equation is homogeneous in f_1 and only depends on the spatial position $\mathbf{x}$ through the perturbation f_1 (recall that f_0 only depends on $\mathbf{v}$). Hence the Fourier modes separate.

After Fourier transformation with

$$\hat{f}_1(t, \mathbf{k}, \mathbf{v}) = \int f_1(t, \mathbf{x}, \mathbf{v}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \tag{4.14}$$

$$\hat{\phi}(t, \mathbf{k}) = \int \phi(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \tag{4.15}$$

the linearised Vlasov equation becomes

$$\frac{\partial \hat{f}_1(t, \mathbf{k}, \mathbf{v})}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \hat{f}_1(t, \mathbf{k}, \mathbf{v}) - \hat{\phi}(t, \mathbf{k}) \frac{ei\mathbf{k}}{m} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{4.16}$$

$$-\mathbf{k}^2 \hat{\phi}(t, \mathbf{k}) = -4\pi e \int \hat{f}_1(t, \mathbf{k}, \mathbf{v}) \mathrm{d}\mathbf{v}. \tag{4.17}$$

We can now consider how one mode evolves, so we fix some $\mathbf{k}$, choose the x-axis along $\mathbf{k}$ and drop the explicit dependence on $\mathbf{k}$. We thus write without hat

$$\frac{\partial f_1(t,\mathbf{v})}{\partial t} + ikv_x f_1(t,\mathbf{v}) - i\phi(t)\frac{ke}{m}\frac{\partial f_0(\mathbf{v})}{\partial v_x} = 0, \tag{4.18}$$

$$k^2\phi(t) = 4\pi e \int f_1(t,\mathbf{v})\mathrm{d}\mathbf{v}. \tag{4.19}$$

Then we apply the Laplace transformation in time to solve the problem. As we will later see in section 4.5, the equations can be written as Volterra equation using Duhamel's principle which explains why a Laplace transformation works. Denote the Laplace transform as

$$\tilde{f}_1(p,\mathbf{v}) = \int_0^\infty f_1(t,\mathbf{v})e^{-pt}\mathrm{d}t, \tag{4.20}$$

$$\tilde{\phi}(p) = \int_0^\infty \phi(t)e^{-pt}\mathrm{d}t. \tag{4.21}$$

Integrating the linearised mode equations yields

$$-f_{in}(\mathbf{v}) + p\tilde{f}_1(p,\mathbf{v}) + ikv_x\tilde{f}_1(p,\mathbf{v}) - i\tilde{\phi}(p)\frac{ke}{m}\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{4.22}$$

$$k^2\tilde{\phi}(p) = 4\pi e \int \tilde{f}_1(p,\mathbf{v})\mathrm{d}\mathbf{v} \tag{4.23}$$

where $f_{in}(\mathbf{v}) = f_1(0,\mathbf{v})$ is the initial datum. Solving the first equation for $\tilde{f}_1(p,\mathbf{v})$ gives

$$\tilde{f}_1(p,\mathbf{v}) = \frac{f_{in}(\mathbf{v}) + i\tilde{\phi}(p)\frac{ke}{m}\frac{\partial f_0(\mathbf{v})}{\partial v_x}}{p+ikv_x}. \tag{4.24}$$

For $\tilde{\phi}(p)$ we find

$$\tilde{\phi}(p) = \frac{4\pi e}{k^2}\left(\int \frac{f_{in}(\mathbf{v})}{p+ikv_x}\mathrm{d}\mathbf{v} + \frac{ike}{m}\tilde{\phi}(p)\int \frac{\partial f_0(\mathbf{v})}{\partial v_x}\frac{\mathrm{d}\mathbf{v}}{p+ikv_x}\right). \tag{4.25}$$

Hence

$$\tilde{\phi}(p) = \frac{4\pi e}{k^2}\frac{\int \frac{f_{in}(\mathbf{v})}{p+ikv_x}\mathrm{d}\mathbf{v}}{1 - i\frac{4\pi e^2}{k^2 m}\int \frac{\partial f_0(\mathbf{v})}{\partial v_x}\frac{\mathrm{d}\mathbf{v}}{p+ikv_x}}. \tag{4.26}$$

Solving the trivial integration in y and z direction, we find

$$\tilde{\phi}(p) = \frac{4\pi e}{k^2}\frac{\int_{-\infty}^{\infty} \frac{g_{in}(u)}{p+iku}\mathrm{d}u}{1 - \frac{\omega_p^2}{k}\int_{-\infty}^{\infty}\frac{\mathrm{d}g_0(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{ku-ip}} \tag{4.27}$$

where with unperturbed density $n = \int f_0(\mathbf{v})\mathrm{d}\mathbf{v}$

$$g_{in}(u) = \int f_{in}(u,v_y,v_z)\mathrm{d}v_y\mathrm{d}v_z, \tag{4.28}$$

$$g_0(u) = \frac{1}{n}\int f_0(u,v_y,v_z)\mathrm{d}v_y\mathrm{d}v_z. \tag{4.29}$$

Proceeding formally as Landau we use the complex inversion formula to find

$$\phi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\tilde{\phi}(p)e^{pt}\mathrm{d}p \tag{4.30}$$

where σ is a large enough positive number.

His idea is to use that $\tilde{\phi}(p)$ is analytic in some right half plane and thus can be continued analytically to the complex plane with possible poles $p_1, p_2, \ldots$. Then we move the contour γ of integration as shown in fig. 1. Written explicitly where $p_1, p_2, \ldots p_k$ are the poles with $\sigma_0 < \Re(p_i) < \sigma$ and $-M < \Im(p_i) < M$ we have

$$\phi(t) = \frac{1}{2\pi i}\left(\int_{-i\infty+\sigma}^{-iM+\sigma} + \int_{-iM+\sigma}^{-iM+\sigma_0} + \int_{-iM+\sigma_0}^{iM+\sigma_0} + \int_{iM+\sigma_0}^{iM+\sigma} + \int_{iM+\sigma}^{i\infty+\sigma}\right)\tilde{\phi}(p)e^{pt}\mathrm{d}p + \sum_{i=1}^{k} e^{p_i t}\mathrm{Res}_{p_i}\tilde{\phi}. \tag{4.31}$$

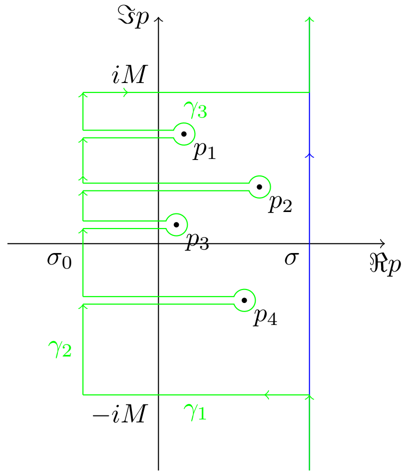

Figure 1 – Deformed contour γ for the integral in p defining ϕ by inverse Laplace transformation. γ_1 and γ_3 are the horizontal paths and γ_2 is the vertical part with $\Re p = \sigma_0$. The picture has been adapted from [23, Chapter 4].

As we take $M \to \infty$, we suppose that the contribution along the line $\Im(p) = \sigma$ vanishes as we suppose that the integral is convergent. We further suppose that the contribution from the horizontal paths γ_1 and γ_3 vanishes. Hence

$$\phi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma_0}^{i\infty+\sigma_0} \tilde{\phi}(p)e^{pt}\mathrm{d}t + \sum_i e^{p_i t}\mathrm{Res}_{p_i}\tilde{\phi}. \tag{4.32}$$

Therefore the long time behaviour is dominated by a contribution growing with $e^{p_k t}$ where p_k is the pole with largest real part.

Already if g_{in} is integrable or square integrable, the integral $\int_{-\infty}^{\infty} \frac{g_{in}(u)}{p+iku}\mathrm{d}u$ is analytic in the region with $\Re(p) > 0$. For p with $\Re(p) \leq 0$ it can easily be continued as analytic function by changing the contour γ with Landau's prescriptions as shown in fig. 2.

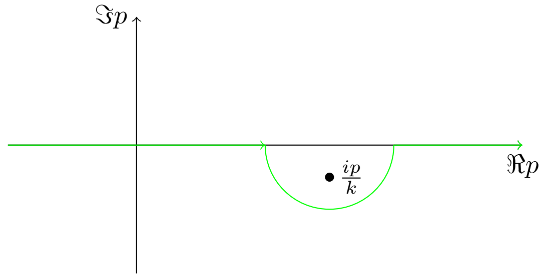

Figure 2 – Landau's prescription for changing the contour of the integral over $g_{in}/(p + iku)$ in $u \in \mathbb{R}$ if $\Re p \leq 0$. The figure has been adapted from [14].

The integral in the denominator can be analytically continued in the same way, so that the only remaining poles are solutions of

$$1 - \frac{\omega_p^2}{k} \int \frac{\mathrm{d}g_0(u)}{\mathrm{d}u} \frac{\mathrm{d}u}{ku - ip} = 0 \tag{4.33}$$

where the u integral is to be done with Landau's prescription. This is beside the contour prescription the dispersion relation eq. (4.10) found earlier, since the Laplace variable p corresponds to $-i\omega$ where ω is the frequency.

From this point of view it becomes more convenient to consider the Laplace transformation as function of $s = ip/k$ which takes a right half plane into an upper half plane. The long time frequency ω therefore is the solution with largest $\Im(\omega)$ of

$$k^2 = Z(s) \tag{4.34}$$

where $s = \omega/k$ and

$$Z(s) = \omega_p^2 \int \frac{\mathrm{d}g_0(u)}{\mathrm{d}u} \frac{\mathrm{d}u}{u - s}. \tag{4.35}$$

Having solved $\tilde{\phi}$, we can find the solution of $\tilde{f}_1$ which has a pole at ikv_x. Hence f_1 also has a term $f_{in}(\mathbf{v})e^{ikv_x t}$. So there is no damping if we consider the velocity distribution. This illustrates how the damping of the electric field without dissipation happens. The full distribution f_1 does not decay but oscillates more quickly at later times. As the electric potential involves the averaging $\int f_1(t, \mathbf{v})\mathrm{d}\mathbf{v}$, it is decaying by the Riemann-Lebesque lemma if f_{in} is integrable.

The rigorous treatment needs to address the various convergence issues. In particular as Backus [1] pointed out, the condition that a function f has an analytic Laplace transformation for the right half plane $\Re(p) \geq \sigma_0$ is not sufficient to conclude that f grows at most by $e^{\sigma_0 t}$.

For this we can consider the example from Widder [29] with the function $f : \mathbb{R}^+ \to \mathbb{R}$ given by

$$f(t) = e^t \sin(e^t). \tag{4.36}$$

For p with $\Re(p) > 1$ the Laplace integral is absolutely convergent and thus defines an analytic function for $\Re(p) > 1$. By substituting $x = e^t$ we find

$$\tilde{f}(p) := \int_0^\infty e^{-pt} f(t)\mathrm{d}t = \int_1^\infty \frac{\sin(x)}{x^p}\mathrm{d}x. \tag{4.37}$$

By partial integration we find for $\Re(p) > 1$

$$\tilde{f}(p) = \left[-\frac{\cos(x)}{x^p}\right]_1^\infty - p\int_1^\infty \frac{\cos(x)}{x^{p+1}}\mathrm{d}x = \cos(1) - p\int_1^\infty \frac{\cos x}{x^{p+1}}\mathrm{d}x. \tag{4.38}$$

However, the RHS also defines an analytic function for $\Re(p) > 0$ and we can continue to conclude that f has a Laplace transformation which is an entire function.

This shows that convergence really can be a problem. In our case it has been solved in [1]. Beside this the main assumptions are that the distribution have a Fourier transformation and that they have a Laplace transformation, i.e. are exponentially bounded.

4.4 Penrose Criterion for Stability

In 1960, Penrose [20] used the argument principle to formulate a criterion in order to know when exponentially growing modes exist, i.e. when there exist real $k > 0$ and frequencies ω with $\Im(\omega) > 0$ such that $k^2 = Z(\omega/k)$ where the integral can be taken along the real line as $\Im(\omega) > 0$. In this case already the naive normal mode analysis shows that perturbations can be unstable. If no such solution exists, he quoted an advanced result on the solution of the Volterra equation to show that all sufficiently regular perturbations are stable. In the same year Backus [1] published his rigorous treatment of Landau's discussion where he showed that the stability is determined by the solutions of $k^2 = Z(\omega/k)$ for $\Im(\omega) > 0$ and the limit $\Im(\omega) \to 0$. Furthermore, he discusses the required stability for this conclusions.

For several kinds of particles the dispersion function Z generalises through the previous discussion as

$$Z(s) = \sum_\alpha \omega_{p\alpha}^2 \int \frac{\mathrm{d}g_{0\alpha}(u)}{\mathrm{d}u} \frac{\mathrm{d}u}{u-s} \tag{4.39}$$

where the (unperturbed) densities are $n_\alpha = \int f_{0\alpha}(\mathbf{v})\mathrm{d}\mathbf{v}$, the (unperturbed) plasma frequencies are

$$\omega_{p\alpha} = \sqrt{\frac{4\pi n_\alpha e_\alpha^2}{m_\alpha}} \tag{4.40}$$

and

$$g_{0\alpha}(u) = \frac{1}{n_\alpha} \int f_{0\alpha}(u, v_y, v_z)\mathrm{d}v_y \mathrm{d}v_z. \tag{4.41}$$

Hence we introduce

$$h(u) = \omega_p^2 g_0(u) \quad \text{respectively} \quad h(u) = \sum_\alpha \omega_{p\alpha}^2 g_{0\alpha}(u) \tag{4.42}$$

and we can write

$$Z(s) = \int \frac{\mathrm{d}h(u)}{\mathrm{d}u} \frac{\mathrm{d}u}{u-s}. \tag{4.43}$$

A solution $k^2 = Z(\omega/k)$ for real $k > 0$ and complex ω with $\Im(\omega) > 0$ exists iff there exists $s \in \mathbb{C}$ with $\Im(s) > 0$ such that $Z(s)$ is a positive real number. If we have a solution $k^2 = Z(\omega/k)$, then $s = \omega/k$ is such a complex number s. Conversely, let $k = \sqrt{Z(s)}$ and $\omega = s/k$. Hence we need to find a criterion when $Z(s)$ takes a positive real value for s in the upper half plane.

For the discussion of Z we introduce the operator $\mathcal{H}$ (cf. [1]) by

$$(\mathcal{H}f)(s) = \int_{-\infty}^{\infty} \frac{f(u)}{u-s} \mathrm{d}u. \tag{4.44}$$

Inside the upper half plane we have from [20, 1] the following lemma.

Lemma 4.1. *If $f \in L^1$ or L^2 then $(\mathcal{H}f)(s)$ is an analytic function of s in the upper half plane.*

Proof. Since u is real, for s in the upper half plane $u - s \neq 0$, so the integrand has derivative $\frac{f(u)}{(u-s)^2}$.

If $f \in L^1$ we can bound

$$\int_{-\infty}^{\infty} \left| \frac{f(u)}{(u-s)^2} \right| \mathrm{d}u \leq \frac{\|f\|_1}{\Im(s)^2} \tag{4.45}$$

or if $f \in L^2$ by Schwarz inequality

$$\int_{-\infty}^{\infty} \left| \frac{f(u)}{(u-s)^2} \right| \mathrm{d}u \leq \|f\|_2 \left(\int_{-\infty}^{\infty} \frac{1}{(u-s)^4} \mathrm{d}u \right)^{1/2}. \tag{4.46}$$

Hence, $\mathcal{H}f$ is differentiable with derivative $\int_{-\infty}^{\infty} \frac{f(u)}{(u-s)^2} \mathrm{d}u$. □

Like Penrose we can characterize the boundary behaviour by the following lemma:

Lemma 4.2. *If $f \in L^1$ and f is Lipschitz continuous, then $(\mathcal{H}f)(s)$ is a continuous and bounded function of s with*

$$(\mathcal{H}f)(x+iy) \to \mathrm{PV} \int_{-\infty}^{\infty} \frac{f(u)}{u-x} \mathrm{d}u + i\pi f(x) \tag{4.47}$$

uniformly over x as $y \to 0+0$ where PV *denotes the principle value and for $s \in \mathbb{C}$ with $\Im(s) \geq 0$*

$$\lim_{|s|\to\infty} (\mathcal{H}f)(s) = 0. \tag{4.48}$$

Proof. We recognise the first part as Plemelj formula which we prove in section 5.5 as corollary 5.25. If $\Im(s) \to \infty$ the second statement is obvious as

$$\left| \int_{\infty}^{\infty} \frac{f(u)}{u-s} \mathrm{d}u \right| \leq \frac{\|f\|_1}{\Im(s)}. \tag{4.49}$$

Otherwise by uniform continuity of the first part we can assume $\Im(s) \geq \delta > 0$. Since f is Lipschitz continuous and $f \in L^1$, it is bounded and $f \in L^2$. Hence by Cauchy-Schwarz

$$\int_{|u| \geq M} \frac{f(u)}{u-s} \mathrm{d}u \to 0 \tag{4.50}$$

as $M \to \infty$. For a fixed M the remaining part is bounded for large enough $\Re(s)$, since

$$\int_{|u| < M} \left| \frac{f(u)}{u-s} \right| \mathrm{d}u \leq \frac{\|f\|_1}{|\Re(s)| - M} \to 0 \tag{4.51}$$

as $\Re(s) \to \infty$, which proves the claim. □

Penrose [20] only assumes $f \in L^2$ and just quoted the result from [16, Chapter 3] where the limit is stated for finite intervals. Backus [1, Lemmata 4 and 5] states the convergence under relaxed regularity (also $f \in L^2$) and different modes of convergences. This kind of operator is also called Hilbert operator which is discussed more generally in [25, Chapter 4].

Hence if we have perturbations around a distribution with g_0 differentiable with $g_0' \in L^1$ and g_0' Lipschitz continuous, then $Z(s)$ is a bounded analytic function in the upper half plane continuous up to the boundary and vanishing at infinity with

$$Z(x + i0) = \mathrm{PV} \int_{-\infty}^{\infty} \frac{h'(u)}{u-x} \mathrm{d}u + i\pi h'(x). \tag{4.52}$$

By the argument principle a value Z_0 not on $Z(x + i0)$ is taken from $Z(s)$ for $\Im(s) > 0$ iff the curve $Z(x+i0)$ has positive winding number around the value Z_0 as illustrated in figs. 3 to 5 which we calculated using Mathematica.

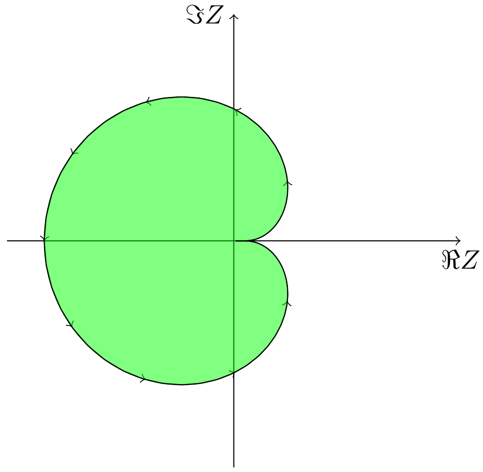

Figure 3 – Curve of $Z(x + i0)$ for $h(u) = e^{-u^2/2}$ (Maxwell distribution, plotted in fig. 6). The traced out area is shaded green and is the image of the upper half plane under Z. We can see that no positive real number is in the image and thus no exponentially growing modes exist.

A point on the positive real axis can only have a non-trivial winding number if $Z(x + i0)$ crosses the real axis. Hence there exists a point on the real axis with positive winding number iff $Z(x + i0)$ crosses the positive real axis from below. Since $\Im(Z(x + i0)) = \pi h'(x)$, the curve $Z(x + i0)$ crosses the real axis from below at the parameter x iff h has a minimum at x. Therefore the full criterion can be formulated as

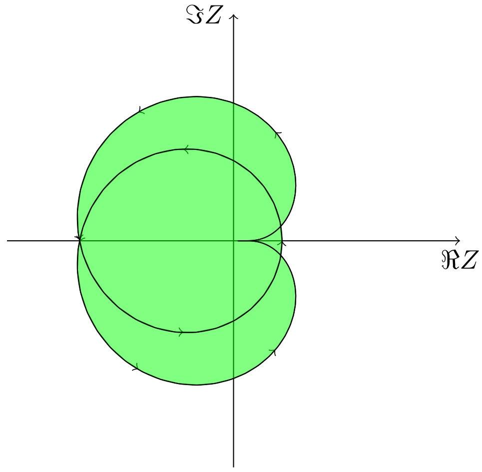

Figure 4 – Curve of $Z(x+i0)$ for $h(u) = e^{-(u-a)^2/2} + e^{-(u+a)^2/2}$ with $a = 1.5$ (Two overlaying Maxwell distributions, plotted in fig. 6). The traced out area is shaded green and is the image of the upper half plane under Z. We can see that Z can take positive real values and the distribution is thus unstable.

Theorem 4.3. *If $h' \in L^1$ and h' is Lipschitz continuous, then exponentially growing modes exist iff there exists $x \in \mathbb{R}$ such that h has a minimum at x and*

$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{h'(u)}{u-x} \mathrm{d}u > 0. \tag{4.53}$$

In the case of a flat minimum of h the condition must be adapted accordingly. Finally, noting as [20] that at a minimum x the derivative $h'(x) = 0$ vanishes, we can write the integral using partial integration as

$$\begin{aligned}\mathrm{PV} \int_{-\infty}^{\infty} \frac{h'(u)}{u-x} \mathrm{d}u &= \lim_{\epsilon \to 0} \left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty} \right) \frac{h'(u) - h'(x)}{u-x} \mathrm{d}u \\ &= \lim_{\epsilon \to 0} \frac{2h(x) - h(x-\epsilon) - h(x+\epsilon)}{\epsilon} + \mathrm{PV} \int_{-\infty}^{\infty} \frac{h(u)-h(x)}{(u-x)^2} \mathrm{d}u \\ &= \int_{-\infty}^{\infty} \frac{h(u)-h(x)}{(u-x)^2} \mathrm{d}u\end{aligned} \tag{4.54}$$

where we can drop the principle value in the last integral, since x is a minimum of h.

4.5 Penrose's Proof of Stability

Having shown when growing modes exist, does not show when a solution is stable. In his paper Penrose used the advanced Paley-Wiener theorem from Fourier analysis to show stability when no growing modes exist. For this he rejected Landau's argument with analytic continuation, even though the proof of the theorem he uses needs analytic continuation.

We characterise the size of a perturbation by the created electric potential $\phi \sim \int f_1(t, \mathbf{v}) \mathrm{d}\mathbf{v}$. As Backus [1] we impose that for a physical realizable solution the electric potential or charge density $\int f_1(t, \mathbf{v}) \mathrm{d}\mathbf{v}$ at wave vector $\mathbf{k}$ should initially be finite and independent of the way of counting, i.e. f_1 should be absolutely integrable. This will be further discussed in section 4.6.

Furthermore we only consider stable configurations such that $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ is absolutely integrable over $\mathbf{v}$ and $h', h'' \in L^2$, where $h(u) = (w_p^2/n) \int f_0(u, v_y, v_z) \mathrm{d}v_y \mathrm{d}v_z$. The absolute integrability of $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ also implies $h' \in L^1$. If we also assume that h'' is bounded, then h' is Lipschitz continuous and we can use the previous argument to show when growing modes exist.

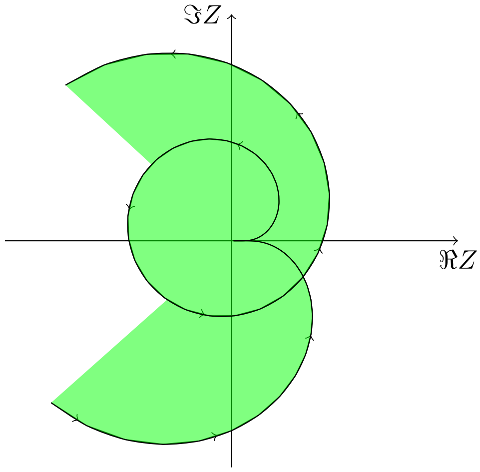

Figure 5 – Curve of $Z(x+i0)$ for $h(u) = 2e^{-u^2} + e^{-(u+a)^2}$ with $a = 2.5$ (Two overlaying Maxwell distributions, plotted in fig. 6). The traced out area is shaded green and is the image of the upper half plane under Z. We can see that Z can take positive real values and the distribution is thus unstable. This configuration is taken from [15].

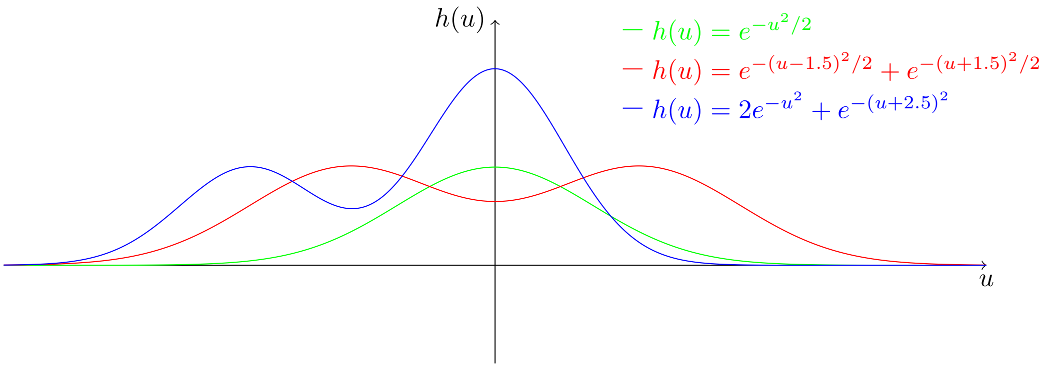

Figure 6 – Marginal background distributions whose stability is discussed with the argument principle in figs. 3 to 5.

Starting from the mode equation

$$\frac{\partial f_1(t,\mathbf{v})}{\partial t} + ikv_x f_1(t,\mathbf{v}) = i\phi(t)\frac{ke}{m}\frac{\partial f_0(\mathbf{v})}{\partial v_x} \tag{4.55}$$

the homogeneous equation

$$\frac{\partial f_1(t,\mathbf{v})}{\partial t} + ikv_x f_1(t,\mathbf{v}) = 0 \tag{4.56}$$

is solved by $e^{-ikv_xt}f_1(0,\mathbf{v})$. Hence by Duhamel's principle (cf. section 5.3) the solution is

$$f_1(t,\mathbf{v}) = e^{-ikv_xt}f_{in}(\mathbf{v}) + \int_0^t \frac{4\pi e^2 i}{mk} e^{-ikv_x(t-s)}\frac{\partial f_0(\mathbf{v})}{\partial v_x}\int f_1(s,\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}s \tag{4.57}$$

where we used $k^2\phi(t) = 4\pi e \int f_1(t,\mathbf{v})\mathrm{d}\mathbf{v}$.

Then we integrate over $\mathbf{v}$ and by the assumed regularity we can use Fubini to find for the potential

$$\phi(t) = \frac{4\pi e}{k^2}\int_{-\infty}^{\infty} e^{-ikut}g_{in}(u)\mathrm{d}u + \frac{i}{k}\int_0^t \int_{-\infty}^{\infty} e^{-iku(t-s)}h'(u)\mathrm{d}u\,\phi(s)\mathrm{d}s \tag{4.58}$$

where $g_{in}(u) = \int f_{in}(u, v_y, v_z)\mathrm{d}v_y \mathrm{d}v_z$. This can be written as Volterra equation

$$\phi(t) + \int_0^t q(t-s)\phi(s)\mathrm{d}s = \psi(t) \tag{4.59}$$

where the forcing is

$$\psi(t) = \frac{4\pi e}{k^2} \int_{-\infty}^{\infty} e^{-ikut} g_{in}(u)\mathrm{d}u \tag{4.60}$$

and the kernel is

$$q(t) = \frac{-i}{k} \int_{-\infty}^{\infty} e^{-ikut} h'(u)\mathrm{d}u. \tag{4.61}$$

From the assumed regularity, ψ is bounded. Also from Fourier analysis we can use lemma 5.1, that if $h', h'' \in L^2$ then the Fourier transformation of h' is absolutely integrable, to conclude $q \in L^1$.

By the theory of the Volterra equation (cf. theorem 5.17) there is a unique locally integrable solution which is given by

$$\phi(t) = \psi(t) - \int_0^t \Gamma(t-s)\psi(s)\mathrm{d}s \tag{4.62}$$

with resolvent kernel $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ uniquely determined by

$$\Gamma(t) + \int_0^t \Gamma(t-s)q(s)\mathrm{d}s = q(t). \tag{4.63}$$

Then by Paley-Wiener theorem 5.18, the resolvent kernel Γ is absolutely integrable iff

$$\tilde{q}(p) := \int_0^\infty q(t)e^{-pt}\mathrm{d}t \neq -1 \tag{4.64}$$

for all $p \in \mathbb{C}$ with $\Re p \geq 0$. In our case the Laplace transformation can be simplified by Fubini's theorem as $h' \in L^1$ to

$$\begin{aligned}\tilde{q}(p) &= \int_0^\infty \frac{-i}{k} \int e^{-ikut} h'(u)\mathrm{d}u\, e^{-pt}\mathrm{d}t \\ &= \frac{-i}{k} \int \left[\frac{e^{-(iku+p)t}}{-(iku+p)}\right]_0^\infty h'(u)\mathrm{d}u \\ &= -\int \frac{h'(u)}{k^2(u - ip/k)}\mathrm{d}u \\ &= -\frac{1}{k^2} Z\left(\frac{ip}{k}\right).\end{aligned} \tag{4.65}$$

So the condition is $k^2 \neq Z(ip/k)$. This is precisely the condition that $Z(s)$ does not take a positive value for $s \in \mathbb{C}$ and $\Im s \geq 0$.

Finally if $\Gamma \in L^1$, then

$$|\phi(t)| \leq |\psi(t)| + \|\Gamma\|_1 \|\psi\|_\infty \tag{4.66}$$

which is bounded. Thus if we characterise the size of a perturbation by the electric potential ϕ, perturbations are stable if $Z(s)$ does not take a positive real value for $\Im s \geq 0$. Hence in the case of the assumed regularity the Penrose criterion is up to the boundary case $\Im s = 0$ a sufficient and necessary condition for stability.

Using the linearity, the Volterra equation 4.59 for the potential ϕ holds accordingly for a plasma with several kinds of particles and thus the same stability argument works.

Reviewing the stability proof we see that we only needed very mild assumptions on the perturbation f_1. We just needed integrability $f_{in} \in L^1$ and $f_1 \in L^1([0,T] \times \mathbb{R}^3)$ for finite T to find a closed Volterra equation for ϕ. This already shows that ϕ is locally integrable and the unique solution can be given with the resolvent kernel Γ.

$$\phi(t) = \psi(t) - \int_0^t \Gamma(t-s)\psi(s)\mathrm{d}s. \tag{4.67}$$

As the forcing is bounded by

$$\left| \frac{4\pi e}{k^2} \int e^{-ikut} g_{in}(u)\mathrm{d}u \right| \leq \frac{4\pi e}{k^2} \left\| f_{in} \right\|_1 , \tag{4.68}$$

the integrability of the resolvent kernel already shows that ϕ is bounded.

4.6 Backus Rigorous Treatment

In the same year 1960, Backus [1] addressed the issues with Landau's treatment, by first showing an exponential bound and uniqueness for the perturbations, which allows the use of Laplace transformation. This also shows that the discussed Volterra equation in the previous section for the electric potential ϕ (cf. eq. (4.59)) only has the considered solution.

He continued by showing that the inverse Laplace transformation is justified and hence the existence of a solution. By proving absolute convergence, he can move the contour of the inverse Laplace transformation in the right half plane to show the stability or instability depending on the existence of zeros of $k^2 - Z(s)$ for s in the upper half plane (cf. eqs. (4.34) and (4.35)) with a more detailed discussion of the required regularity. Further, he noted that under the finer measure of the perturbation size $\int \left| \int f_1(u, v_y, v_z)\mathrm{d}v_y \mathrm{d}v_z \right| \mathrm{d}u$ and quite relaxed regularity conditions, there exists growing perturbations even for the Maxwell distribution. This again shows that Landau damping crucially depends on how we define the size of a perturbation.

Finally, he investigated the scope of the linearised theory for predicting the asymptotic behaviour of the non-linear system, which is limited as the neglected terms are not decaying but growing. For the case of a thermonuclear plasma he estimated that after only 220μs the linearised theory is no longer admissible.

Since these calculations are quite technically, we will only discuss the exponential bound in a slightly different way than [1] to take the perpendicular directions to $\mathbf{k}$ into account.

As Backus [1] pointed out, an exponential bound is non-trivial for a linear equation in time, as we can see from the counterexample that for the backward heat equation

$$4\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial x^2} = 0 \tag{4.69}$$

we have for $x \in \mathbb{R}$ and $t \in [0, T]$ the solution

$$(T-t)^{-1/2} \exp(x^2/t - T). \tag{4.70}$$

We start again with the mode equation where we replaced ϕ

$$\frac{\partial f_1(t, \mathbf{v})}{\partial t} + ikv_x f_1(t, \mathbf{v}) = \frac{4\pi e^2}{m}\frac{i}{k} \int f_1(t, \mathbf{w})\mathrm{d}\mathbf{w} \frac{\partial f_0(\mathbf{v})}{\partial v_x}. \tag{4.71}$$

We suppose again that $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ is absolutely integrable and measure the size of a perturbation by the charge density in this mode

$$Q(t) := \int f_1(t, \mathbf{v})\mathrm{d}\mathbf{v} \tag{4.72}$$

which should be independent of the way of counting, i.e.

$$A(t) := \int |f_1(t, \mathbf{v})| \, \mathrm{d}\mathbf{v} < \infty. \tag{4.73}$$

We will only consider perturbations for which there exists a time interval $[0, T]$ over which A is finite, as we suppose that every physical realisable perturbation has this form. A priori we still allow $A \to \infty$ in finite time.

As in the previous section (eq. (4.57)) the equivalent integral equation by Duhamel's principle is

$$f_1(t, \mathbf{v}) = e^{-ikv_x t} f_{in}(\mathbf{v}) + \int_0^t \frac{4\pi e^2 i}{mk} e^{-ikv_x(t-s)} \frac{\partial f_0(\mathbf{v})}{\partial v_x} Q(s) \mathrm{d}s \tag{4.74}$$

for which we prove the following lemma.

Lemma 4.4. *If f_1 is such that, $A(t)$ is bounded for $t \in [0, T]$ and for all $t \in [0, T]$ satisfies*

$$f_1(t, \mathbf{v}) = e^{-ikv_x t} f_{in}(\mathbf{v}) + \int_0^t \frac{4\pi e^2 i}{mk} e^{-ikv_x(t-s)} \frac{\partial f_0(\mathbf{v})}{\partial v_x} Q(s) \mathrm{d}s \tag{4.75}$$

then for $t \in [0, T]$

$$|Q(t)| \leq A(0) \cosh(\omega_p t) \tag{4.76}$$

and Q is continuous with respect to t. Furthermore, for $\mathbf{v}$ where $f_{in}(\mathbf{v}) := f_1(0, \mathbf{v})$ and $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ are defined (which are almost all $\mathbf{v}$)

$$|f_1(t, \mathbf{v})| \leq |f_{in}(\mathbf{v})| + A(0) \frac{\omega_p}{nk} \left| \frac{\partial f_0(\mathbf{v})}{\partial v_x} \right| \sinh(\omega_p t) \tag{4.77}$$

and $\frac{\partial f_1(\mathbf{v})}{\partial t}$ exists, is continuous and satisfies

$$\frac{\partial f_1(t, \mathbf{v})}{\partial t} + ikv_x f_1(t, \mathbf{v}) = \frac{4\pi e^2}{m} \frac{i}{k} \int f_1(t, \mathbf{w}) \mathrm{d}\mathbf{w} \frac{\partial f_0(\mathbf{v})}{\partial v_x}. \tag{4.78}$$

Note that the boundedness of $A(t)$ in particular implies that f_{in} is absolutely integrable.

Proof. By the assumptions we can integrate eq. (4.75) over $\mathbf{v}$ and use Fubini to find

$$Q(t) = \int e^{-ikv_x t} f_{in}(\mathbf{v}) \mathrm{d}\mathbf{v} + \int_{u=-\infty}^{\infty} \int_{s=0}^{t} \frac{\omega_p^2 i}{k} e^{-iku(t-s)} g_0'(u) Q(s) \mathrm{d}s \mathrm{d}u. \tag{4.79}$$

By integration by parts with respect to u and s the second term is

$$\begin{aligned}\int_{u=-\infty}^{\infty} \int_{s=0}^{t} \frac{\omega_p^2 i}{k} e^{-iku(t-s)} g_0'(u) Q(s) \mathrm{d}s \mathrm{d}u &= -\int_{u=-\infty}^{\infty} \int_{s=0}^{t} (t-s) \omega_p^2 e^{-iku(t-s)} g_0(u) Q(s) \mathrm{d}s \mathrm{d}u \\ &= -\omega_p^2 \int_{s_1=0}^{t} \int_{s_2=0}^{s_1} Q(s_2) \int_{-\infty}^{\infty} e^{-iku(t-s)} g_0(u) \mathrm{d}u \mathrm{d}s_2 \mathrm{d}s_1.\end{aligned} \tag{4.80}$$

Putting the second term back gives

$$Q(t) = \int e^{-ikv_x t} f_{in}(\mathbf{v}) \mathrm{d}\mathbf{v} - \omega_p^2 \int_{s_1=0}^{t} \int_{s_2=0}^{s_1} Q(s_2) \int_{-\infty}^{\infty} e^{-iku(t-s)} g_0(u) \mathrm{d}u \mathrm{d}s_2 \mathrm{d}s_1. \tag{4.81}$$

The density g_0 is non-negative and normalised by $\int_{\infty}^{\infty} g_0(u) \mathrm{d}u = 1$, so that

$$\left| \int_{-\infty}^{\infty} e^{-iku(t-s)} g_0(u) \mathrm{d}u \right| \leq \int_{-\infty}^{\infty} |g_0(u)| \, \mathrm{d}u = 1. \tag{4.82}$$

Hence we can bound

$$|Q(t)| \leq A(0) + \omega_p^2 F(t) \tag{4.83}$$

where

$$F(t) = \int_0^t \int_0^{s_1} |Q(s_2)| \, \mathrm{d}s_2 \mathrm{d}s_1. \tag{4.84}$$

Let $h(t) = F''(t) - \omega_p^2 F(t)$, then as $F''(t) = |Q(t)|$, we can conclude $h(t) \leq A(0)$.

On the other hand

$$F(t) = \int_0^t \int_0^{s_1} h(s_2) e^{\omega_p(2s_1 - s_2 - t)} \mathrm{d}s_2 \mathrm{d}s_1 \tag{4.85}$$

which can be verified by integration by parts. Plugging in $h(t) \leq A(0)$, we find

$$\begin{aligned} F(t) &\leq A(0) \int_0^t \int_0^{s_1} e^{\omega_p(2s_1 - s_2 - t)} \mathrm{d}s_2 \mathrm{d}s_1 \\ &= \frac{A(0)}{\omega_p} \int_0^t \left(e^{\omega_p(2s_1 - t)} - e^{\omega_p(s_1 - t)} \right) \mathrm{d}s_1 \\ &= \frac{A(0)}{\omega_p^2} \left(\frac{e^{\omega_p t} - e^{-\omega_p t}}{2} - (1 - e^{-\omega_p t}) \right) \\ &= \frac{A(0)}{\omega_p^2} \left(\cosh(\omega_p t) - 1 \right). \end{aligned} \tag{4.86}$$

Hence by eq. (4.83), the first bound eq. (4.76) is proven. With this bound eq. (4.81) implies that Q is continuous with respect to t.

Plugging this bound into the integral equation eq. (4.75) gives for $\mathbf{v}$ such that $f_{in}(\mathbf{v}) := f_1(0, \mathbf{v})$ and $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ are defined

$$\begin{aligned} |f_1(t, \mathbf{v})| &\leq |f_{in}(\mathbf{v})| + \int_0^t \frac{\omega_p^2}{nk} \left| \frac{\partial f_0(\mathbf{v})}{\partial v_x} \right| A(0) \cosh(\omega_p t) \mathrm{d}s \\ &= |f_{in}(\mathbf{v})| + A(0) \frac{\omega_p}{nk} \left| \frac{\partial f_0(\mathbf{v})}{\partial v_x} \right| \sinh(\omega_p t). \end{aligned} \tag{4.87}$$

Finally by continuity of Q the derivative $\frac{\partial f_1(t,\mathbf{v})}{\partial t}$ exists by eq. (4.75) and is given by

$$\frac{\partial f_1(t, \mathbf{v})}{\partial t} = -ikv_x e^{-ikv_x t} f_{in}(\mathbf{v}) + \frac{i\omega_p^2}{nk} \frac{\partial f_0(\mathbf{v})}{\partial v_x} Q(t) + \int_0^t \frac{\omega_p^2}{n} e^{-ikv_x(t-s)} \frac{\partial f_0(\mathbf{v})}{\partial v_x} Q(s) \mathrm{d}s \tag{4.88}$$

which is continuous and satisfies eq. (4.78). □

By linearity this shows the uniqueness of the solution, because the difference of two solutions with the same initial data is a solution with $A(0) = 0$.

Hence no disturbance can grow at a rate faster than ω_p. Therefore, f_1 and so ϕ are exponentially bounded unless the solution cannot be extended beyond T. By the Laplace transformation we can however show the existence of such a solution [1], so that f_1 and ϕ are exponentially bounded with rate ω_p. Moreover, with the assumed regularity of the unperturbed system that $\frac{\partial f_0(\mathbf{v})}{\partial v_x}$ is absolutely integrable, also $A(t)$ grows at most with rate ω_p which justifies the assumed integrability in Penrose's argument.

For his stability analysis he only considered the marginal distribution

$$g_1(t, u) = \int f_1(t, u, v_y, v_z) \mathrm{d}v_y \mathrm{d}v_z \tag{4.89}$$

and its evolution. We consider a stable distribution

$$g_0(t, u) = n^{-1} \int f_0(t, u, v_y, v_z) \mathrm{d}v_y \mathrm{d}v_z \tag{4.90}$$

such that $g_0' \in L^1 \cap L^2$ and satisfying the stability criterion, i.e. for all $\omega \in \mathbb{C}$ with $\Im(\omega) > 0$

$$\left| k^2 - Z\left(\frac{\omega}{k}\right) \right| \geq \kappa > 0. \tag{4.91}$$

For example we could consider the Maxwell distribution. His first stability result [1, Theorem 2] is for a perturbation g_1 with initial datum g_{in}.

Theorem 4.5. *If $g_{in} \in L^1 \cap L^2$, then $A(t) = \int |g_1(t,u)| \, \mathrm{d}u$ is bounded for all time and if $g_0'(u)$ is bounded, $\int |g_1(t,u)|^2 \, \mathrm{d}u$ is also bounded for all time.*

However, with this stricter measure A for the size of a perturbation he finds [1, Theorem 3].

Theorem 4.6. *There exists $g_{in} \in L^1$ such that $A(t) = \int |g_1(t,u)| \, \mathrm{d}u$ is not a bounded function of time.*

Note that our previous stability argument by Penrose shows that $\int g_1(t,u)\mathrm{d}u$ is still bounded. For his results he used

$$g_{in}(u) = \begin{cases} (u-\omega)^{-1} \left|\log(u-\omega)\right|^{-3/2} & \text{if } \omega < u < \omega + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4.92}$$

where $\omega \in \mathbb{R}$ is such that $g_0'(\omega) \neq 0$ and brings this to a contradiction with the assumption that $A(t)$ is bounded.

4.7 A More Quantitative Approach by Mouhot and Villani

For their work on non-linear Landau damping, Mouhot and Villani [17, Chapter 3] tried to find a more quantitive statement about linear Landau damping, which we are going to show in this section.

For their treatment we use periodic boundary conditions, i.e. we consider the space to be the d-dimensional torus $\mathbb{T}_L^d = \mathbb{R}^d / L\mathbb{Z}^d$, which introduces a length scale L, and do not restrict to Coulomb interaction, but rather allow a general interaction potential[3] $\psi_{12}(\mathbf{x},\mathbf{y}) = W(\mathbf{x}-\mathbf{y})$. Following the same arguments from section 3.6 we find the Vlasov equation as mean field limit

$$\frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} = 0 \tag{4.93}$$

$$\mathbf{F} = -\frac{\partial (W * \rho)(t,\mathbf{x})}{\partial \mathbf{x}} = -\nabla W * \rho \tag{4.94}$$

where $*$ denotes the convolution on $\mathbb{T}_L^d$ and $\rho(t,\mathbf{x}) = \int f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$.

It should be noted, that the reduction of the interaction potential to a torus is not obviously possible, because for the Coulomb interaction or Newtonian gravity the flat space interaction is not integrable for dimensions $d \geq 3$. In the case of Coulomb interaction, however, we can appeal to Debye screening to introduce a cut-off such that the potential $W_f : \mathbb{R}^d \to \mathbb{R}$ can be made periodic by

$$W(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} W_f(\mathbf{x} + L\mathbf{n}). \tag{4.95}$$

An overview of the periodization issue can be found in their paper [17, Chapter 2].

In the case of a torus, we get discrete Fourier modes for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$. For such a $\mathbf{k}$ we have the Fourier coefficient

$$\hat{W}(\mathbf{k}) = \int_{\mathbb{T}_L^d} W(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} W_f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} \mathrm{d}\mathbf{x} = \hat{W}_f(\mathbf{k}) \tag{4.96}$$

where $\hat{W}_f(\mathbf{k})$ is the Fourier transformation of W_f over $\mathbb{R}^d$.

For the stability discussion we will assume that W is even, i.e. $W(-\mathbf{x}) = W(\mathbf{x})$, and that W, $\nabla W \in L^1(\mathbb{T}_L^d)$. Further we rescale W such that we can set $m = 1$. By the same argument (cf. section 4.1) we linearise around a fixed distribution $f_0(\mathbf{v})$. For a perturbation $f_1(t,\mathbf{x},\mathbf{v})$ the linearised equation is like before

$$\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - (\nabla W * \rho)(t,\mathbf{x}) \cdot \frac{\partial f_0(\mathbf{x})}{\partial \mathbf{v}} = 0 \tag{4.97}$$

[3] This allows for example to treat galaxies under gravity by the same analysis where the stars are particles.

where $\rho(t,\mathbf{x}) = \int f_1(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$.

As in Penrose's stability proof we use Duhamel's principle (cf. section 5.3) to find an integral integration. The homogeneous equation

$$\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} = 0 \tag{4.98}$$

has the solution $f_1(0,\mathbf{x}-t\mathbf{v},\mathbf{v})$ so the equivalent integral equation is

$$f_1(t,\mathbf{x},\mathbf{v}) = f_{in}(\mathbf{x}-t\mathbf{v},\mathbf{v}) + \int_0^t (\nabla W * \rho)(s,\mathbf{x}-(t-s)\mathbf{v})\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\mathrm{d}s \tag{4.99}$$

where $f_{in}(\mathbf{x},\mathbf{v}) = f_1(0,\mathbf{x},\mathbf{v})$ is the initial datum.

As before we assume that the solution f_1 is integrable over finite time. Integrating over $\mathbf{v}$ gives a closed equation for ρ

$$\rho(t,\mathbf{v}) = \rho_{in}(\mathbf{x}-t\mathbf{v}) + \int_0^t\int_{\mathbb{R}^d} (\nabla W * \rho)(s,\mathbf{x}-(t-s)\mathbf{v})\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\mathrm{d}\mathbf{v}\mathrm{d}s \tag{4.100}$$

where $\rho_{in}(\mathbf{x}) = \int f_{in}(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$ is the initial datum and we can justify the use of Fubini, because for any finite time where $\rho(t,\cdot)$ is bounded, the convolution is bounded as we assume $\nabla W \in L^1$.

For the stability analysis we consider analytic stable configurations $f_0(\mathbf{v})$. In particular this implies integrability and we can look at the evolution of a Fourier coefficient for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n}\in\mathbb{Z}^d$ over space and the continuous modes over velocity.

For the different terms we find

$$\int_{\mathbb{T}_L^d} \rho(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} = \hat{\rho}(t,\mathbf{k}) \tag{4.101}$$

and by a simple change of variables

$$\int_{\mathbb{T}_L^d} \rho_{in}(\mathbf{x}-t\mathbf{v})e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} = \int_{\mathbb{T}_L^d}\int_{\mathbb{R}^d} f_{in}(\mathbf{x},\mathbf{v})e^{-i\mathbf{k}\cdot(\mathbf{x}+t\mathbf{v})}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} = \hat{f}_{in}(\mathbf{k},t\mathbf{k}). \tag{4.102}$$

By the same pattern the last term is

$$\begin{aligned}
&\int_{\mathbb{T}_L^d}\int_0^t\int_{\mathbb{R}^d} (\nabla W * \rho)(s,\mathbf{x}-(t-s)\mathbf{v})\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\mathrm{d}\mathbf{v}\mathrm{d}s\, e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}\\
&\qquad = \int_0^t\int_{\mathbb{T}_L^d}\int_{\mathbb{R}^d} (\nabla W * \rho)(s,\mathbf{x})\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}e^{-i\mathbf{k}\cdot(\mathbf{x}+(t-s)\mathbf{v})}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}\mathrm{d}s\\
&\qquad = \int_0^t (\widehat{\nabla W * \rho})(s,\mathbf{k})\cdot i\mathbf{k}(t-s)\hat{f}_0((t-s)\mathbf{k})\mathrm{d}s\\
&\qquad = \int_0^t -\left|\mathbf{k}\right|^2(t-s)\hat{W}(\mathbf{k})\hat{f}_0((t-s)\mathbf{k})\hat{\rho}(s,\mathbf{k})\mathrm{d}s.
\end{aligned} \tag{4.103}$$

Putting it together we find the Volterra equation for $\hat{\rho}(t,\mathbf{k})$

$$\rho(t,\mathbf{k}) + \int_0^t K^0(t-s,\mathbf{k})\rho(s,\mathbf{k})\mathrm{d}s = a(t,\mathbf{k}) \tag{4.104}$$

where the kernel is

$$K^0(t,\mathbf{k}) = \hat{W}(\mathbf{k})\hat{f}_0(t\mathbf{k})\left|\mathbf{k}\right|^2 t \tag{4.105}$$

and the forcing is

$$a(t,\mathbf{k}) = \hat{f}_{in}(\mathbf{k},t\mathbf{k}). \tag{4.106}$$

We suppose f_0 is analytic so that we can find constants C_0, $\lambda > 0$ such that for all $\eta\in\mathbb{R}^d$

$$\left|\hat{f}_0(\eta)\right| \le C_0 e^{-\lambda|\eta|}. \tag{4.107}$$

Since we have a discrete spectrum either $\mathbf{k} = \mathbf{0}$ or $|\mathbf{k}| \geq (2\pi/L)$. In the first case $K^0(t, \mathbf{k} = \mathbf{0})$ is vanishing. Otherwise for $t \geq 0$

$$\left|K^0(t, \mathbf{k})\right| \leq \left|\hat{W}(\mathbf{k})\right| C_0 \left|\mathbf{k}\right|^2 t e^{-\lambda t |\mathbf{k}|}. \tag{4.108}$$

Hence for $p \in \mathbb{C}$ with $\Re(p) > -\lambda \left|\mathbf{k}\right|$ we have an absolutely convergent Laplace transformation

$$\tilde{K}^0(p, \mathbf{k}) = \int_0^\infty K^0(t, \mathbf{k}) e^{-pt} \mathrm{d}t. \tag{4.109}$$

We can now prove a quantative statement about stability (cf. [17, Lemma 3.6]).

Lemma 4.7. *Given a stable configuation $f_0 = f_0(\mathbf{v})$ and constants C_0, λ such that for all $\eta \in \mathbb{R}^d$*

$$\left|\hat{f}_0(\eta)\right| \leq C_0 e^{-\lambda |\eta|} \tag{4.110}$$

and an even interaction potential $W : \mathbb{T}_L^d \to \mathbb{R}$ with $\|W\|_{L^1(\mathbb{T}_L^d)} \leq C_W$. Further suppose that there exist $\kappa > 0$ such that for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and all $p \in \mathbb{C}$ with $\Re(p) > -\lambda \left|\mathbf{k}\right|$

$$\left|\tilde{K}^0(p, \mathbf{k}) + 1\right| \geq \kappa \tag{4.111}$$

holds where

$$\tilde{K}^0(p, \mathbf{k}) = \int_0^\infty K^0(t, \mathbf{k}) e^{-pt} \mathrm{d}t, \tag{4.112}$$

$$K^0(t, \mathbf{k}) = \hat{W}(\mathbf{k}) \hat{f}_0(t\mathbf{k}) \left|\mathbf{k}\right|^2 t. \tag{4.113}$$

Then any solution $\phi(t, \mathbf{k})$ of

$$\phi(t, \mathbf{k}) + \int_0^t K^0(t - s, \mathbf{k}) \phi(s, \mathbf{k}) = a(t, \mathbf{k}) \tag{4.114}$$

satisfies for any $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and any $\lambda' < \lambda$

$$\sup_{t \geq 0} |\phi(t, \mathbf{k})| \, e^{\lambda' |\mathbf{k}| t} \leq \left(1 + \frac{C_0 C_W}{\sqrt{8} (\lambda - \lambda')^2 \kappa}\right) \sup_{t \geq 0} |a(t, \mathbf{k})| \, e^{\lambda |\mathbf{k}| t}. \tag{4.115}$$

Proof. If $\sup_{t \geq 0} |a(t, \mathbf{k})| \, e^{\lambda |\mathbf{k}| t} = \infty$, there is nothing to prove. Likewise if $\mathbf{k} = \mathbf{0}$, then $K^0(t, \mathbf{k} = \mathbf{0})$ vanishes and the statement becomes obvious. Hence we only need to consider the case $A(\mathbf{k}) := \sup |a(t, \mathbf{k})| \, e^{\lambda |\mathbf{k}| t} < \infty$ and $|\mathbf{k}| \geq 2\pi/L$. We show the inequality for each $\mathbf{k}$ separately and thus consider $\mathbf{k}$ as given constant in the rest of the proof.

Noting as in theorem 5.11 and lemma 5.16 we can multiply the functions by a factor $e^{\lambda' |\mathbf{k}| t}$ for some constant λ' to get an equivalent equation, i.e. if we let

$$\phi_{\lambda'}(t, \mathbf{k}) = \phi(t, \mathbf{k}) e^{\lambda' |\mathbf{k}| t}, \quad K^0_{\lambda'}(t, \mathbf{k}) = K^0(t, \mathbf{k}) e^{\lambda' |\mathbf{k}| t} \text{ and } a_{\lambda'}(t, \mathbf{k}) = a(t, \mathbf{k}) e^{\lambda' |\mathbf{k}| t} \tag{4.116}$$

then the Volterra equation eq. (4.114) is equivalent to

$$\phi_{\lambda'}(t, \mathbf{k}) + \int_0^t K^0_{\lambda'}(t - s, \mathbf{k}) \phi_{\lambda'}(s, \mathbf{k}) = a_{\lambda'}(t, \mathbf{k}) \tag{4.117}$$

by multiplying the equation by $e^{\lambda' |\mathbf{k}| t}$. Further we can bound for $\lambda' < \lambda$

$$|a'_\lambda(t, \mathbf{k})| < A(\mathbf{k}) e^{-(\lambda - \lambda') |\mathbf{k}| t} \tag{4.118}$$

and since $\|W\|_{L^1} \leq C_W$ implies $\left|\hat{W}(\mathbf{k})\right| \leq C_W$, the kernel is bounded by

$$\left|K^0_{\lambda'}(t, \mathbf{k})\right| \leq C_0 C_W \left|\mathbf{k}\right|^2 t e^{-(\lambda - \lambda') |\mathbf{k}| t}. \tag{4.119}$$

Hence for any $\lambda' < \lambda$, the functions $a_{\lambda'}(t, \mathbf{k})$ and $K^0_{\lambda'}(t, \mathbf{k})$ are integrable and square integrable over time. Also

$$\tilde{K}^0_{\lambda'}(p, \mathbf{k}) = \tilde{K}^0(p - \lambda' |\mathbf{k}|, \mathbf{k}) \tag{4.120}$$

so that by Paley-Wiener theorem 5.18 the resolvent $r_{\lambda'}(t, \mathbf{k})$ given by

$$r_{\lambda'}(t, \mathbf{k}) + \int_0^t r_{\lambda'}(t-s, \mathbf{k}) K^0_{\lambda'}(s, \mathbf{k}) \mathrm{d}t = K^0_{\lambda'}(t, \mathbf{k}). \tag{4.121}$$

is integrable over time t iff for all $p \in \mathbb{C}$ with $\Re(p) \geq -\lambda' |\mathbf{k}|$ holds

$$\tilde{K}^0(p, \mathbf{k}) \neq -1. \tag{4.122}$$

By theorem 5.17 the solution ϕ is given with the resolvent kernel by

$$\phi_{\lambda'}(t, \mathbf{k}) = a_{\lambda'}(t, \mathbf{k}) - \int_0^t r_{\lambda'}(t-s, \mathbf{k}) a_{\lambda'}(s, \mathbf{k}) \mathrm{d}s. \tag{4.123}$$

Hence if $r_{\lambda'}(t, \mathbf{k})$ is integrable and $a_{\lambda'}(t, k)$ is bounded over t, also $\phi_{\lambda'}(t, \mathbf{k})$ is bounded over t.

Hence for any $\lambda' < \lambda$ also $\phi_{\lambda'}(t, \mathbf{k})$ is integrable and square integrable over t, because there exists $\epsilon > 0$ such that $\phi_{\lambda'+\epsilon}(t, \mathbf{k})$ is bounded over t and $|\phi_{\lambda'}(t, \mathbf{k})| \leq |\phi_{\lambda'+\epsilon}(t, \mathbf{k})| e^{-\epsilon |\mathbf{k}| t}$.

Now extend $a_{\lambda'}$, $\phi_{\lambda'}$ and $K^0_{\lambda'}$ to functions of $\mathbb{R}$ by setting them zero for $t < 0$. Then the Volterra equation can be written as equation for $t \in \mathbb{R}$

$$\phi_{\lambda'}(t, \mathbf{k}) + \int_{\mathbb{R}} K^0_{\lambda'}(t-s, \mathbf{k}) \phi_{\lambda'}(s, \mathbf{k}) \mathrm{d}s = a_{\lambda'}(t, \mathbf{k}). \tag{4.124}$$

Since $\phi_{\lambda'}$, $K^0_{\lambda'}$ and $a_{\lambda'}$ are integrable and square integrable over time, we can take the Fourier transformation in the time variable, i.e.

$$\hat{\phi}_{\lambda'}(\omega, \mathbf{k}) = \int \phi_{\lambda'}(t, \mathbf{k}) e^{-i\omega t} \mathrm{d}t \tag{4.125}$$

and accordingly for $K^0_{\lambda'}$ and $a_{\lambda'}$, where the transformed functions are square integrable and bounded. The integral equation becomes equivalently

$$\hat{\phi}_{\lambda'}(\omega, \mathbf{k}) + \hat{K}^0_{\lambda'}(\omega, \mathbf{k}) \hat{\phi}_{\lambda'}(\omega, \mathbf{k}) = \hat{a}_{\lambda'}(\omega, \mathbf{k}). \tag{4.126}$$

As $\hat{K}^0_{\lambda'}(\omega, \mathbf{k}) = \tilde{K}^0(i\omega + \lambda' |\mathbf{k}|, \mathbf{k}) \neq -1$ this is equivalent to

$$\hat{\phi}_{\lambda'}(\omega, \mathbf{k}) = \frac{\hat{a}_{\lambda'}(\omega, \mathbf{k})}{1 + \hat{K}^0_{\lambda'}(\omega, \mathbf{k})}. \tag{4.127}$$

The denominator is at least κ using the assumption $\left| \tilde{K}^0(p, \mathbf{k}) + 1 \right| \geq \kappa$ for $p = \lambda' |\mathbf{k}| + i\omega$. Hence by Parsevals's theorem

$$\|\phi_{\lambda'}(t, \mathbf{k})\|_{L^2(\mathrm{d}t)} \leq \frac{1}{\kappa} \|a_{\lambda'}(t, \mathbf{k})\|_{L^2(\mathrm{d}t)}. \tag{4.128}$$

Using Cauchy-Schwarz we find

$$\left| \int_0^t K^0_{\lambda'}(t-s, \mathbf{k}) \phi_{\lambda'}(s, \mathbf{k}) \mathrm{d}s \right| \leq \left\| K^0_{\lambda'}(t, \mathbf{k}) \right\|_{L^2(\mathrm{d}t)} \|\phi_{\lambda'}(t, \mathbf{k})\|_{L^2(\mathrm{d}t)}. \tag{4.129}$$

Hence we get from the Volterra equation eq. (4.117)

$$|\phi_{\lambda'}(t, \mathbf{k})| \leq |a_{\lambda'}(t, \mathbf{k})| + \left\| K^0_{\lambda'}(t, \mathbf{k}) \right\|_{L^2(\mathrm{d}t)} \|\phi_{\lambda'}(t, \mathbf{k})\|_{L^2(\mathrm{d}t)}. \tag{4.130}$$

Taking the supremum we find as $|a_{\lambda'}(t, \mathbf{k})| \leq |a_{\lambda}(t, \mathbf{k})|$ for $\lambda' \leq \lambda$

$$\|\phi_{\lambda'}(t, \mathbf{k})\|_{L^\infty(\mathrm{d}t)} \leq \|a_{\lambda}(t, \mathbf{k})\|_{L^\infty(\mathrm{d}t)} + \frac{\|a_{\lambda'}(t, \mathbf{k})\|_{L^2(\mathrm{d}t)} \left\| K^0_{\lambda'}(t, \mathbf{k}) \right\|_{L^2(\mathrm{d}t)}}{\kappa}. \tag{4.131}$$

The kernel can be bounded by eq. (4.119)

$$\left\|K^0_{\lambda'}(t,\mathbf{k})\right\|_{L^2(\mathrm{d}t)} \leq C_0 C_W \left|\mathbf{k}\right|^2 \left[\int_0^\infty t^2 e^{-2(\lambda-\lambda')|\mathbf{k}|t}\mathrm{d}t\right]^{1/2} = \frac{C_0 C_W \left|\mathbf{k}\right|^{1/2}}{2(\lambda-\lambda')^{3/2}}. \tag{4.132}$$

By Cauchy-Schwarz we can also bound as $\lambda' < \lambda$

$$\left\|a_{\lambda'}(t,\mathbf{k})\right\|_{L^2(\mathrm{d}t)} = \left(\int_0^\infty \left|a_\lambda(t,\mathbf{k})\right|^2 e^{-2(\lambda-\lambda')|\mathbf{k}|t}\right)^{1/2} \leq \frac{\left\|a_\lambda(t,\mathbf{k})\right\|_{L^\infty(\mathrm{d}t)}}{\left(2(\lambda-\lambda')\left|\mathbf{k}\right|\right)^{1/2}}. \tag{4.133}$$

Plugging in these bounds into eq. (4.130) gives the desired estimate. □

Putting this together we arrive at their quantitative statement [17, Theorem 3.1].

Theorem 4.8 (Quantitative Linear Landau Damping)**.** *Given a stable configuation $f_0 = f_0(\mathbf{v})$ and constants C_0, λ such that for all $\eta \in \mathbb{R}^d$*

$$\left|\hat{f}_0(\eta)\right| \leq C_0 e^{-\lambda|\eta|} \tag{4.134}$$

and an even interaction potential $W : \mathbb{T}^d_L \to \mathbb{R}$ with $\|W\|_{L^1(\mathbb{T}^d_L)} \leq C_W$ and $\|\nabla W\|_{L^1(\mathbb{T}^d_L)} \leq C_W$. Further suppose that there exist $\kappa > 0$ such that for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and all $p \in \mathbb{C}$ with $\Re(p) > -\lambda\left|\mathbf{k}\right|$

$$\left|\tilde{K}^0(p,\mathbf{k}) + 1\right| \geq \kappa \tag{4.135}$$

holds where $\tilde{K}^0$ is from lemma 4.7.

Given initial data $f_{in} = f_{in}(\mathbf{x},\mathbf{v})$ and positive constants α, C_{in} such that for all $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}$ and $\eta \in \mathbb{R}^d$

$$\left|\hat{f}_{in}(\mathbf{k},\eta)\right| \leq C_{in} e^{-\alpha|\eta|}. \tag{4.136}$$

Then the solution $f_1 = f_1(t,\mathbf{x},\mathbf{v})$ of the linearised Vlasov equation eq. (4.97) with initial datum f_{in} converges as $t \to \infty$ weakly to $f_\infty = \langle f_{in}\rangle$ defined by

$$f_\infty = \frac{1}{L^d}\int_{\mathbb{T}^d_L} f_{in}(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x} \tag{4.137}$$

and $\rho(\mathbf{x}) = \int_{\mathbb{R}^d} f_1(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$ converges strongly to

$$\rho_\infty = \frac{1}{L^d}\int_{\mathbb{T}^d_L}\int_{\mathbb{R}^d} f_{in}(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}. \tag{4.138}$$

More quantitative for any $\lambda' < \min(\lambda,\alpha)$ and all $r \in \mathbb{N}$ and $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and $\eta \in \mathbb{R}^d$

$$\|\rho(t,\cdot) - \rho_\infty\|_{C^r} = O(e^{-2\pi\lambda' t/L}) \tag{4.139}$$

$$\left|\hat{f}(t,\mathbf{k},\eta) - \hat{f}_\infty(\mathbf{k},\eta)\right| = O\left(e^{-\lambda'|\mathbf{k}|t}\right). \tag{4.140}$$

Proof. By the remarks and the lemma before for $\lambda' < \min(\lambda,\alpha)$ and $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$

$$\left|\hat{\rho}(t,\mathbf{k})\right| \leq \left(1 + \frac{C_0 C_W}{\sqrt{8}(\lambda-\lambda')^2\kappa}\right) C_{in} e^{-\lambda|\mathbf{k}|t} = C_\rho e^{-\lambda|\mathbf{k}|t} \tag{4.141}$$

where C_ρ is a constant and we used that the forcing is $\hat{f}_{in}(\mathbf{k},t\mathbf{k})$ which we assumed to be sufficiently bounded (cf. eq. (4.106)). In particular for $\mathbf{k} \neq 0$ and any $\lambda'' < \lambda'$ and $t \geq 1$ as $\left|\mathbf{k}\right| \geq (2\pi/L)$

$$\left|\hat{\rho}(t,\mathbf{k})\right| = O\left(e^{-2\pi\lambda'' t/L} e^{-(\lambda'-\lambda'')|\mathbf{k}|t}\right). \tag{4.142}$$

Hence any Sobolev norm of $\rho - \rho_\infty$ converges to zero like $O(e^{-2\pi\lambda'' t/L})$, where we can choose λ'' arbitrary close to λ. By Sobolov embedding theorem [28] the same is true for any C^r norm.

With a bound for the density we can return to the Volterra equation 4.99. As for the density we can take the Fourier transformation in space and time. By the same pattern we find for $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n} \in \mathbb{Z}^d$ and $\eta \in \mathbb{R}^d$

$$\hat{f}_1(t, \mathbf{k}, \eta) = \hat{f}_{in}(\mathbf{k}, \eta + t\mathbf{k}) + \int_0^t \widehat{\nabla W}(\mathbf{k})\hat{\rho}(s, \mathbf{k}) \cdot \widehat{\nabla_{\mathbf{v}} f_0}(\eta + (t-s)\mathbf{k})\mathrm{d}s. \tag{4.143}$$

Hence

$$\hat{f}_1(t, \mathbf{k}, \eta - t\mathbf{k}) = \hat{f}_{in}(\mathbf{k}, \eta) + \int_0^t \widehat{\nabla W}(\mathbf{k})\hat{\rho}(s, \mathbf{k}) \cdot \widehat{\nabla_{\mathbf{v}} f_0}(\eta - s\mathbf{k})\mathrm{d}s \tag{4.144}$$

and setting $\mathbf{k} = 0$ as $\widehat{\nabla W}(0) = 0$

$$\hat{f}_1(t, 0, \eta) = \hat{f}_{in}(0, \eta), \tag{4.145}$$

which shows that $\langle f \rangle = L^{-d} \int_{\mathbb{T}_L^d} f \mathrm{d}\mathbf{x}$ is constant in time. For $|\mathbf{k}| \geq (2\pi/L)$ and putting in the bounds

$$\begin{aligned}
\left|\hat{f}(t, \mathbf{k}, \eta - t\mathbf{k})\right| &\leq \left|\hat{f}_{in}(\mathbf{k}, \eta)\right| + \int_0^t \left|\widehat{\nabla W}(\mathbf{k})\right| |\hat{\rho}(s, \mathbf{k})| \left|\widehat{\nabla_v f_0}(\eta - s\mathbf{k})\right| \mathrm{d}s \\
&\leq C_{in} e^{-\alpha|\eta|} + \int_0^t C_W C_\rho e^{-\lambda'|\mathbf{k}|s} |\eta - s\mathbf{k}| \, C_0 e^{-\lambda|\eta - s\mathbf{k}|}\mathrm{d}s \\
&\leq C_f \left(e^{-\alpha|\eta|} + \int_0^t e^{-\lambda'|\mathbf{k}|s} e^{-\left(\frac{\lambda+\lambda'}{2}\right)|\eta - s\mathbf{k}|}\mathrm{d}s\right)
\end{aligned} \tag{4.146}$$

where we used that $\lambda' < \frac{\lambda'+\lambda}{2} < \lambda$ and C_f is a constant. The integral can be estimated as

$$\begin{aligned}
\int_0^t e^{-\lambda'|\mathbf{k}|s} e^{-\left(\frac{\lambda+\lambda'}{2}\right)|\eta - s\mathbf{k}|}\mathrm{d}s &\leq \int_0^t e^{-\lambda'|\eta|} e^{-\left(\frac{\lambda-\lambda'}{2}\right)|\eta - s\mathbf{k}|}\mathrm{d}s \\
&\leq \frac{L}{\pi(\lambda - \lambda')} e^{-\left(\lambda' - \frac{\lambda-\lambda'}{2}\right)|\eta|}.
\end{aligned} \tag{4.147}$$

With $\lambda'' = \lambda' - \frac{\lambda-\lambda'}{2}$ we can put this back to arrive at

$$\left|\hat{f}_1(t, \mathbf{k}, \eta - t\mathbf{k})\right| \leq C e^{-\lambda''|\eta|} \tag{4.148}$$

for some constant C. For fixed η and $\mathbf{k} \neq 0$

$$\left|\hat{f}_1(t, \mathbf{k}, \eta)\right| \leq C e^{-\lambda''|\eta + t\mathbf{k}|} = O\left(e^{-\lambda''|\mathbf{k}|t}\right). \tag{4.149}$$

Hence pointwise $\hat{f}_1$ converges exponentially fast to the Fourier transform of $\langle f_{in} \rangle$. As λ' and so λ'' can be arbitrary close to $\min(\alpha, \lambda)$, this proves the claim. □

Like before we can then relate the assumption $\left|\tilde{K}_0(p, \mathbf{k}) + 1\right| \leq \kappa$ to the boundary behaviour and recover Penrose's stability criterion.

4.8 Interpretation of Results

We only considered perturbations following the linearised Vlasov equation which is not time-reversible. However, we hope that this also describes the behaviour for the full Vlasov equation which is time-reversible. This may raise the first objection why we only observe the exponentially decaying modes and not the growing modes which should exist by time reversion.

Mouhot and Villani [17] answered this question for the full Vlasov equation by phase-mixing, i.e. mathematically we only observe the evolution of the weak topology of a solution. This corresponds to the observation of Landau in his first paper [14] that the electric field is decaying but not the full distribution $f_1 = f_1(t, \mathbf{x}, \mathbf{v})$ and is already present in the linearised case.

The idea of phase mixing can already be illustrated in one dimension on a torus for the free transport equation by fig. 7. Even though the local density in phase space is only transported and does not converge, the average density in any volume converges to the average density. A physical example illustration by [21] is the idea of an oil film which partly covers a glass of water under stirring.

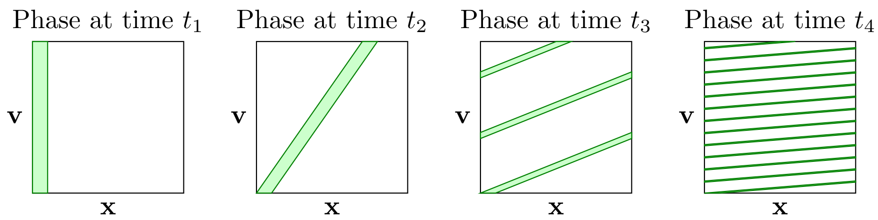

Figure 7 – Illustration of phase mixing in one dimension: Consider the free transport equation $\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} = 0$ and the space as torus. At time t_1 all particles are collected in a strip whose evolution to later times $t_2 < t_3 < t_4$ is shown. We can see that the spatial marginal distribution becomes more and more uniform.

Already in 1967 it was noted in [10] that this is not strong convergence which is observable through plasma wave echos. They predicted if we create a wave in a plasma and let it damp away and then create a second wave, it will still be damped, but we can also observe a spontaneous third wave, the so called echo.

Finally, a common interpretation is the so called surfer picture that particles slower than the wave are accelerated while particles faster are decelerated. Since for a Maxwell distribution there are more faster particles than slower particles this is supposed to show damping. However, the statement that slower particles are accelerated in general is wrong as a more careful study by [8] and [23, Chapter 4] shows.

5 Mathematical Analysis

In this section we develop the mathematical theory which we used to prove stability. We start with the Fourier and Laplace transformation which are used throughout the essay. Afterwards we introduce Duhamel's principle which allows to express the Vlasov equation as a Volterra equation. We then develop the theory of the Volterra equation ending with the Paley-Wiener theorem. Finally, we show the Plemelj formula which we need to use the argument principle in Penrose stability criterion.

5.1 Fourier Transformation

For a function $f : \mathbb{R}^d \to \mathbb{C}$ introduce the Fourier transformation

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x} \tag{5.1}$$

and the inverse Fourier transformation

$$f(\mathbf{x}) = (2\pi)^{-d} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k}. \tag{5.2}$$

From linearity of the integral, both transformations are linear.

For $f \in L^2$ Parseval's theorem [25, Chapter 3] states that the integral converges in L^2 and is (up to a factor $\sqrt{2\pi}$) a linear isometry

$$\left\|\hat{f}\right\|_2 = (2\pi)^{-d/2} \|f\|_2 \tag{5.3}$$

and the inversion formula holds.

For a differentiable function f, we can formulate a simplified lemma [20] of [25, Theorem 84] showing integrability of the Fourier transformation using Parseval's theorem. This is used in Penrose stability proof to show that the kernel is integrable.

Lemma 5.1. *If $f : \mathbb{R} \to \mathbb{C}$ is differentiable function and $f, f' \in L^2$, then the Fourier transformation $\hat{f}$ is in L^1.*

Proof. This simpler proof has been suggested by Mouhot. By the Cauchy-Schwarz inequality

$$\begin{aligned}\int_{-\infty}^{\infty} |\hat{f}(k)| \mathrm{d}k &= \int_{-\infty}^{\infty} \frac{(1+|k|)|\hat{f}(k)|}{(1+|k|)} \mathrm{d}k \\ &\leq \left(\int_{-\infty}^{\infty} \frac{1}{(1+|k|)^2} \mathrm{d}k\right)^{1/2} \left(\int_{-\infty}^{\infty} (1+|k|)^2 |\hat{f}(k)|^2\right)^{1/2} \\ &\leq C \left(\int_{-\infty}^{\infty} |\hat{f}(k)|^2 \mathrm{d}k + \int_{-\infty}^{\infty} |k|^2 \, |\hat{f}(k)|^2 \mathrm{d}k\right)\end{aligned} \tag{5.4}$$

where $C = 2(\int_{-\infty}^{\infty} (1+|k|)^{-2} \mathrm{d}k)^{1/2} < \infty$.

Since the Fourier transformation of f' is $ik\hat{f}(k)$, Parseval's theorem shows

$$\left(\int_{-\infty}^{\infty} |\hat{f}(k)|^2 \mathrm{d}k + \int_{-\infty}^{\infty} |k|^2 \, |\hat{f}(k)|^2 \mathrm{d}k\right) = \|f\|_2 + \|f'\|_2 < \infty \tag{5.5}$$

which finished the proof. ☐

For $f \in L^1$ the integral $\hat{f}(\mathbf{k})$ exists for all $\mathbf{k}$ with $\|\hat{f}\|_\infty \leq \|f\|_1$ and by dominated convergence $\hat{f}$ is a continuous function of $\mathbf{k}$.

The inversion is more difficult in this case and often expressed in the Fourier integral theorem (proven in a slightly different form in [25, Theorem 23]).

Theorem 5.2. *Let $f \in L^1(\mathbb{R})$ and if there exists a neighbourhood of x in which f is continuously differentiable then*

$$f(x) = \frac{1}{2\pi} \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} e^{ixu} \int_{\infty}^{\infty} f(t) e^{-iut} \mathrm{d}t \mathrm{d}u. \tag{5.6}$$

For the uniqueness the following theorem [24, Lemma 4.2, Page 87] suffices.

Theorem 5.3. *Let $f \in L^1$ such that its Fourier transform $\hat{f}$ is also in L^1. Then for almost every $\mathbf{x}$*

$$f(\mathbf{x}) = (2\pi)^{-d} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}. \tag{5.7}$$

By linearity this shows the uniqueness of the Fourier transformation, because if $f, g \in L^1$ have almost everywhere the same Fourier transformation, then the difference $f - g$ has almost everywhere vanishing Fourier transformation. Hence by the theorem $f(\mathbf{x}) - g(\mathbf{x}) = 0$ for almost all $\mathbf{x}$.

Proof. Let $f, g \in L^1$, then their Fourier transformation $\hat{f}$ respectively $\hat{g}$ are bounded, so that we can use Fubini to find

$$\int \hat{f}(\mathbf{k}) g(\mathbf{k}) \mathrm{d}\mathbf{k} = \int f(\mathbf{k}) \hat{g}(\mathbf{k}) \mathrm{d}\mathbf{k}. \tag{5.8}$$

Now let g be a modulated Gaussian for $\delta > 0$ and fixed $\mathbf{x} \in \mathbb{R}^d$ given by

$$g(\mathbf{k}) = (2\pi)^{-d} e^{-\pi\delta|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.9}$$

Its Fourier transformation is by elementary calculation

$$\hat{g}(\mathbf{y}) = \int g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{y}} \mathrm{d}\mathbf{k} = (2\pi)^{-d} \delta^{-d/2} e^{-|\mathbf{x}-\mathbf{y}|^2/(4\pi\delta)} \tag{5.10}$$

which we call $K_\delta(\mathbf{x} - \mathbf{y})$. This is a kernel as

$$\int K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} = 1 \tag{5.11}$$

and for every $\eta > 0$,

$$\lim_{\delta \to 0} \int_{|\mathbf{y}| \geq \eta} K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} = 0. \tag{5.12}$$

The relation eq. (5.8) becomes

$$(2\pi)^{-d} \int \hat{f}(\mathbf{k}) e^{-\pi\delta|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k} = \int f(\mathbf{k}) K_\delta(\mathbf{x} - \mathbf{k}) \mathrm{d}\mathbf{k}. \tag{5.13}$$

As $\delta \to 0$ the LHS converges by dominated convergence to $(2\pi)^{-d} \int \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k}$.
By change of variable $\mathbf{k} = \mathbf{x} - \mathbf{y}$ the RHS becomes

$$\int f(\mathbf{x} - \mathbf{y}) K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{5.14}$$

Since K_δ integrates to one we find

$$\int f(\mathbf{x} - \mathbf{y}) K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} - f(\mathbf{x}) = \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})] K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{5.15}$$

With $f_\mathbf{y}(\mathbf{x}) = f(\mathbf{x} - \mathbf{y})$ the L^1 norm of the difference is

$$\int \left| \int [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})] K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} \right| \mathrm{d}\mathbf{x} \leq \int \|f_\mathbf{y} - f\|_1 K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{5.16}$$

By dominated convergence $\|f_\mathbf{y} - f\|_1 \to 0$ as $\mathbf{y} \to 0$, so for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f_\mathbf{y} - f\| < \epsilon$ if $|\mathbf{y}| \leq \delta$. Hence

$$\begin{aligned} \int \|f_\mathbf{y} - f\|_1 K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} &\leq \epsilon + \int_{|\mathbf{y}| > \delta} \|f_\mathbf{y} - f\|_1 K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &\leq \epsilon + 2 \|f\|_1 \int_{|\mathbf{y}| > \delta} K_\delta(\mathbf{y}) \mathrm{d}\mathbf{y}. \end{aligned} \tag{5.17}$$

As $\delta \to 0$ the RHS converges to ϵ. Since this holds for all $\epsilon > 0$, this show that $\int f(\mathbf{k}) K_\delta(\mathbf{x} - \mathbf{k}) \mathrm{d}\mathbf{k}$ converges in L^1 to f. $\square$

A function on a torus $\mathbb{T}_L^d := \mathbb{R}_L^d / L\mathbb{Z}^d$ can be considered as a function $f : [0, L] \to \mathbb{C}$. Since $[0, L]$ has finite measure by Schwarz inequality $L^2([0, L]) \subset L^1([0, L])$. The functions $e^{i\mathbf{k}\cdot\mathbf{x}}$ are only smooth if $\mathbf{k} = (2\pi/L)\mathbf{n}$ for $\mathbf{n} \in \mathbb{Z}^d$ and the theory of Fourier series shows that the functions $e^{i\mathbf{k}\cdot\mathbf{x}}$ with these coefficients are a basis.

More precisely, we define the Fourier series $a_\mathbf{n}$ of f by

$$a_\mathbf{n} = \int_{[0,L]^d} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \mathrm{d}\mathbf{x}, \qquad \text{with } \mathbf{k} = \frac{2\pi}{L}\mathbf{n}. \tag{5.18}$$

From [24, Chapter 4] we note the following similar properties for $f \in L^1([0, L])$

- If $a_{\mathbf{n}} = 0$ for all $\mathbf{n}$, then $f(\mathbf{x}) = 0$ for almost every $\mathbf{x}$,
- $\sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} r^{|\mathbf{n}|} e^{i\mathbf{n}\cdot\mathbf{x}} \to f(\mathbf{x})$ for almost every $\mathbf{x}$ as $r \to 1$, $r < 1$,
- If $f \in L^2$, then Parseval's relation states

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} |a_{\mathbf{n}}|^2 = L^{-1} \int_{[0,L]^d} |f(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}, \tag{5.19}$$

- If $f \in L^2$, then $\sum_{\mathbf{n}\in\mathbb{Z},|\mathbf{n}|\le N} a_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}$ converges in L^2 to f as $N \to \infty$.

Hence we note that we have very similar properties except that we have a discrete spectrum with a minimal non-zero frequency $2\pi/L$ related to the length L. Therefore, we still write for $\mathbf{k} = (2\pi/L)\mathbf{n}$ and $\mathbf{n} \in \mathbb{Z}^d$

$$\hat{f}(\mathbf{k}) = \int_{[0,L]^d} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}. \tag{5.20}$$

5.2 Laplace Transformation

A less common transformation more useful to boundary value problems is the Laplace transformation. Our presentation is following [2, 6].

For the motivation we consider functions $f : \mathbb{R}^+ \to \mathbb{C}$ which are exponentially bounded, i.e. there exist real numbers c and M such that for all $t \in \mathbb{R}^+$

$$|f(t)| \le M e^{ct}. \tag{5.21}$$

For two such functions f and g we can use pointwise addition to obtain a new exponentially bounded function. Using the convolution in $\mathbb{R}^+$

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) \mathrm{d}\tau \tag{5.22}$$

we have a closed multiplication between functions since with $|f(t)| \le M_1 e^{c_1 t}$ and $|g(t)| \le M_2 e^{c_2 t}$ for all t

$$|(f * g)(t)| \le M_1 M_2 \int_0^t e^{c_1 \tau} e^{c_2 (t-\tau)} \mathrm{d}\tau \le M e^{ct} \tag{5.23}$$

where $M = M_1 M_2$ and $c = \max(c_1, c_2)$. We can prove that this indeed defines an algebraic ring which motivates the algebraic operator calculus by Mikusiński whose idea is to solve the problem in the quotient field of the ring [2, 7].

Viewed from this point, the Laplace transformation is a ring isomorphism to a space where convolution as multiplication becomes pointwise multiplication which can easily be inverted. Since we analyse the complex structure of the transformation, we use a direct approach towards Laplace transformation.

Definition 5.4 (Laplace Transformation)**.** *For a function $f : \mathbb{R}^+ \to \mathbb{C}$ define the Laplace transformation $\tilde{f} = \mathcal{L}f$ by*

$$\tilde{f}(p) = \int_0^\infty f(t) e^{-pt} \mathrm{d}t \tag{5.24}$$

whenever the integral converges.

The condition[4] that $f : \mathbb{R}^+ \to \mathbb{C}$ has an exponential bound $|f(t)| \le M e^{ct}$ for all $t \in \mathbb{R}^+$ and some real constants c and M can be slightly generalised to the condition that $f(t)e^{-c_0 t} \in L^1$ for a real constant c_0 by choosing $c_0 > c$.

From this we can show

[4] Backus' treatment (cf. section 4.6) shows the existence of an exponential bound in our application. In the operator calculus of [2] the attention is restricted to this class of exponentially bounded functions. The more general form is used in [6].

- For the right half plane consisting of $p \in \mathbb{C}$ with $\Re(p) \geq c_0$, $\tilde{f}(p)$ is defined and a bounded analytic function of p.
- $\tilde{f}(p) \to 0$ uniformly as $\Re(p) \to \infty$. Since $f(t)e^{-c_0 t} \in L^1$, we can adapt the proof of the Riemann-Lebesque lemma to show that also $\tilde{f}(x+iy) \to 0$ uniformly over $x \in [c_0, \infty)$ as $y \to \infty$. Combine this to find that $\tilde{f}(p) \to 0$ uniformly as $|p| \to \infty$ for $\Re(p) \geq c_0$.

Theorem 5.5 (Convolution under Laplace transformation)**.** *Let f, g be functions of $\mathbb{R}^+$ with a constant c_0 such that $f(t)e^{-c_0 t}$ and $g(t)e^{-c_0 t}$ are in L^1. Then $(f * g)(t)e^{-c_0 t} \in L^1$ and for $p \in \mathbb{C}$ with $\Re(p) \geq c_0$*

$$\mathcal{L}(f * g)(p) = (\mathcal{L}f)(p)(\mathcal{L}g)(p). \tag{5.25}$$

Proof. Let $p \in \mathbb{C}$ with $\Re(p) \geq c_0$. Since $(\mathcal{L}g)(p)$ is a constant we can put it under the integral of $(\mathcal{L}f)(p)$ to find

$$(\mathcal{L}f)(p)(\mathcal{L}g)(p) = \int_0^\infty f(\tau)e^{-p\tau} \int_0^\infty g(s)e^{-ps}\mathrm{d}s\mathrm{d}\tau. \tag{5.26}$$

Since $f(t)e^{-c_0 t} \in L^1$ and $g(t)e^{-c_0 t} \in L^1$ the integral is absolutely convergent. Hence we can substitute $t = s + \tau$ and change the order of integration

$$\begin{aligned}(\mathcal{L}f)(p)(\mathcal{L}g)(p) &= \int_0^\infty f(\tau)e^{-p\tau} \int_t^\infty g(t-\tau)e^{-p(t-\tau)}\mathrm{d}t\mathrm{d}\tau \\ &= \int_0^\infty \left(\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau \right) e^{-pt}\mathrm{d}t \\ &= \mathcal{L}(f * g)(p).\end{aligned} \tag{5.27}$$

Since the integral is absolutely convergent, taking $p = c_0$ shows $(f * g)(t)e^{-c_0 t} \in L^1$. □

Theorem 5.6 (Differentiation under Laplace transformation)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be continuously differentiable function and c_0 be a real constant such that $f(t)e^{-c_0 t} \in L^1$ and $f'(t)e^{-c_0 t} \in L^1$. Then for $p \in \mathbb{C}$ with $\Re(p) \geq c_0$*

$$(\mathcal{L}f')(p) = -f(0) + p(\mathcal{L}f)(p). \tag{5.28}$$

Proof. Using integration by parts

$$\begin{aligned}(\mathcal{L}f')(p) &= \int_0^\infty f'(t)e^{-pt}\mathrm{d}t \\ &= \left[fe^{-pt}\right]_0^\infty + p\int_0^\infty f(t)e^{-pt}\mathrm{d}t.\end{aligned}$$ □

From complex analysis we can prove an inversion formula.

Theorem 5.7 (Complex Inversion Formula)**.** *If $\tilde{f}$ is a bounded, analytic function in some right half plane consisting of p with $\Re(p) \geq c_0$ and $\tilde{f}(p) \to 0$ uniformly as $|p| \to \infty$ for $\Re(p) \geq c_0$ and*

$$\int_{-i\infty+c_0}^{i\infty+c_0} \left|\tilde{f}(p)\right| |\mathrm{d}p| \leq \infty, \tag{5.29}$$

then $\tilde{f}$ is the Laplace transformation of

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty+c_0}^{i\infty+c_0} e^{pt}\tilde{f}(p)\mathrm{d}p. \tag{5.30}$$

Proof. Consider the contour integral along γ as shown in fig. 8.

For p_0 enclosed by γ we have by Cauchy's integral formula

$$\tilde{f}(p_0) = \frac{1}{2\pi i}\int_\gamma \frac{\tilde{f}(p)}{p_0 - p}\mathrm{d}p. \tag{5.31}$$

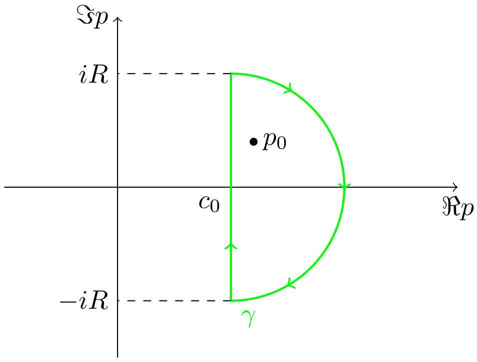

Figure 8 – Contour γ for proving the Laplace inversion theorem

Letting $R \to \infty$ the contribution of the arc vanishes by the assumptions and we find for all $p_0 \in \mathbb{C}$ with $\Re p_0 > c_0$

$$\tilde{f}(p_0) = \frac{1}{2\pi i} \int_{-i\infty+c_0}^{i\infty+c_0} \frac{\tilde{f}(p)}{p_0 - p} \mathrm{d}p. \tag{5.32}$$

On the other hand as $\Re(p_0 - p) > 0$

$$\int_0^\infty e^{-t(p_0-p)} \mathrm{d}t = \frac{1}{p_0 - p}. \tag{5.33}$$

Hence

$$\begin{aligned} \tilde{f}(p_0) &= \frac{1}{2\pi i} \int_{-i\infty+c_0}^{i\infty+c_0} \tilde{f}(p) \int_0^\infty e^{-t(p_0-p)} \mathrm{d}t \mathrm{d}p \\ &= \int_0^\infty e^{-tp_0} \frac{1}{2\pi i} \int_{-i\infty+c_0}^{i\infty+c_0} e^{pt} \tilde{f}(p) \mathrm{d}p \mathrm{d}t \end{aligned} \tag{5.34}$$

where we can use Fubini since the integrand is by assumption absolutely integrable. Identifying f, this proves the theorem. □

Finally we note that the Laplace transformation is unique with respect to the L^1 norm, i.e. identifying functions which are equal almost everywhere. By linearity this is equivalent to the statement that a function f with vanishing Laplace transformation is $\|f\|_1 = 0$, i.e. zero almost everywhere.

A direct proof is given in [6, Chapter 5] by the following lemma.

Lemma 5.8. *Let f be a function $f : \mathbb{R}^+ \to \mathbb{C}$ with Laplace transformation $\tilde{f} = \mathcal{L}f$. If there exist real constants c_0 and $\sigma > 0$ such that $f(t)e^{-c_0 t} \in L^1$ and for all $n \in \mathbb{N}$*

$$\tilde{f}(c_0 + n\sigma) = 0, \tag{5.35}$$

then $\|f\|_1 = 0$.

For the proof we use the following lemma.

Lemma 5.9. *If $f : [a, b] \to \mathbb{C}$ is a continuous function such that*

$$\int_a^b f(x) x^n \mathrm{d}x = 0 \tag{5.36}$$

for all $n \in \mathbb{N}$, then $f(x) = 0$ for all $x \in [a, b]$.

Proof. By Weierstraß approximation theorem, there exists for every $\epsilon > 0$ a polynomial p such that $\sup_{x\in[a,b]} |f(x) - p(x)| < \epsilon$.

Using linearity the assumption implies that $\int_a^b f(x)p(x)\mathrm{d}x = 0$. Hence

$$\int_a^b [f(x)]^2 \mathrm{d}x \le \epsilon \int_a^b |f(x)| \,\mathrm{d}x + \int_a^b f(x)p(x)\mathrm{d}x = \epsilon \|f\|_1 . \tag{5.37}$$

Since ϵ is arbitrary and $\|f\|_1 < \infty$ for a continuous function, $\|f\|_2 = 0$. As f is continuous, this implies that f vanishes everywhere. □

Now we can prove the uniqueness.

Proof of Lemma 5.8. Introduce the continuous function $\Phi : \mathbb{R}^+ \to \mathbb{C}$ by

$$\Phi(t) = \int_0^t e^{-c_0\tau} f(\tau)\mathrm{d}\tau \tag{5.38}$$

which is bounded by $f(t)e^{-c_0 t} \in L^1$ and $\lim_{t\to\infty} \Phi(t) = \tilde{f}(c_0) = 0$.

For $p \in \mathbb{C}$ with $\Re(p) > c_0$ we get by partial integration

$$\tilde{f}(p) = \int_0^\infty e^{-(p-c_0)t} e^{-c_0 t} f(t)\mathrm{d}t = (p - c_0) \int_0^\infty e^{-(p-c_0)t} \Phi(t)\mathrm{d}t. \tag{5.39}$$

Thus for $n = 1, 2, \ldots$, setting $p = c_0 + n\sigma$ gives

$$\int_0^\infty e^{-n\sigma t} \Phi(t)\mathrm{d}t = 0. \tag{5.40}$$

Substituting $x = e^{-\sigma t}$, $t = -\frac{\log x}{\sigma}$ and letting $\psi(x) = \Phi(-\frac{\log(x)}{\sigma})$ this is

$$\int_0^1 x^{n-1} \psi(x)\mathrm{d}x = 0. \tag{5.41}$$

Since $\lim_{x\to 0} \psi(x) = \lim_{t\to\infty} \Phi(t) = 0$, setting $\psi(0) = 0$ makes ψ a continuous function.

Hence by the previous lemma, $\psi(x) = 0$ for all $x \in [0, 1]$ and so Φ is vanishing. Therefore, for all $t \in \mathbb{R}^+$

$$\begin{aligned} 0 &= \int_0^t e^{-c_0\tau} f(\tau)\mathrm{d}\tau \\ &= \left[e^{-c_0\tau} \int_0^\tau f(x)\mathrm{d}x \right]_0^t + \int_0^t c_0 e^{-c_0\tau} \int_0^\tau f(x)\mathrm{d}x\mathrm{d}\tau \\ &= e^{-c_0 t} \int_0^t f(x)\mathrm{d}x + \int_0^t c_0 e^{-c_0\tau} \int_0^\tau f(x)\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{5.42}$$

Since the second term is differentiable with respect to t, the first one is as well and we find $\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(x)\mathrm{d}x = 0$. As $\int_0^0 f(x)\mathrm{d}x = 0$, this implies that $\int_0^t f(x)\mathrm{d}x = 0$ for all t, i.e. $\|f\|_1 = 0$. □

Another view by [2, Section 36] comes from relating the complex inversion formula to the Fourier integral theorem (cf. theorems 5.2 and 5.3) for a function f

$$f(x) = \frac{1}{2\pi} \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{ixu} \int_\infty^\infty f(t)e^{-iut}\mathrm{d}t\mathrm{d}u. \tag{5.43}$$

Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a function such that $f(t)e^{-\sigma t} \in L^1(\mathbb{R}^+)$ with a constant σ. Then extend f to $\mathbb{R}$ by setting $f(x) = 0$ for $x < 0$. Then in terms of the Laplace transformation choosing $p = \sigma + iu$

$$(\mathcal{L}f)(p) = \int_\infty^\infty f(t)e^{-\sigma t} e^{-iut}\mathrm{d}t. \tag{5.44}$$

Hence if the Laplace transformation of f vanishes, we can use theorem 5.3 to conclude that f vanishes almost everywhere.

5.3 Duhamel's Principle

The Duhamel principle rewrites a differential equation as integral equation. For completeness we will briefly repeat the discussion from [26].

For a function u depending on time t and position x consider the problem

$$\partial_t u - Lu = f, \qquad u(0,x) = \phi(x). \tag{5.45}$$

where L is a differential operator and f is a forcing which may depend on time.

Suppose we can solve the homogeneous problem

$$\partial_t u - Lu = 0, \qquad u(0,x) = \phi(x) \tag{5.46}$$

with solution operator S

$$(S(t)\phi)(x) = u(t,x) \tag{5.47}$$

satisfying

$$(S(0)\phi)(x) = \phi(x), \qquad (\partial_t - L)(S(t)\phi) = 0. \tag{5.48}$$

Then by Duhamel's Principle the original problem has the solution

$$u(t,x) = (S(t)\phi)(x) + \int_0^t (S(t-s)f_s)(x)\mathrm{d}s, \tag{5.49}$$

which can easily be verified as $u(0,x) = \phi(x)$ and

$$(\partial_t - L)u(t,x) = 0 + S(t-s)f_s(x)|_{t=s} + \int_0^t \underbrace{(\partial_t - L)(S(t-s)f_s)}_{=0}\mathrm{d}s = f_t(x). \tag{5.50}$$

In general the solution of the integrable equation may not be differentiable in x anymore and is therefore called mild solution. However, the integral equation in our case has a unique solution which should describe the physical evolution.

5.4 Volterra Equation

Start with the integral equation for u with given forcing f and kernel k

$$u(x) + \int_{-\infty}^{\infty} k(x-y)u(y)\mathrm{d}y = f(x) \tag{5.51}$$

which can be formally solved by Fourier transformation which takes the convolution into a pointwise product.

From [25, Theorem 145] a precise form is:

Theorem 5.10. *Let $k \in L^1$, $\|k\|_1 < 1$, $f \in L^2$, and $\hat{k}$, $\hat{f}$ be the Fourier transformation of k respectively f. Then*

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{f}(\omega)}{1+\hat{k}(\omega)} e^{ix\omega}\mathrm{d}w \tag{5.52}$$

is well-defined and the unique solution in L^2 of

$$u(x) + \int_{-\infty}^{\infty} k(x-y)u(y)\mathrm{d}y = f(x). \tag{5.53}$$

The uniqueness is understood as almost everywhere or equivalently with respect to the L^2 norm.

Proof. By Parseval's theorem, Fourier transformation is an isometry in L^2 and by [25, Theorem 65] the integral equation is equivalent to

$$\hat{u}(\omega) + \hat{k}(\omega)\hat{u}(\omega) = \hat{f}(\omega) \tag{5.54}$$

for ω in $\mathbb{R}$, where $\hat{u}, \hat{f}, \hat{k}$ are the Fourier transformations of u, f, k.

Since $\|k\|_1 < 1$, also $\|\hat{k}\|_\infty < 1$ and we can equivalently write

$$\hat{u}(\omega) = \frac{\hat{f}(\omega)}{1 + \hat{k}(\omega)} \tag{5.55}$$

which is in L^2. Hence f is uniquely given by the inverse Fourier transformation. □

Restrict now to functions which are only non-vanishing in $\mathbb{R}^+$, then the integral equation becomes

$$u(x) + \int_0^x k(x-y)u(y)\mathrm{d}y = f(x) \tag{5.56}$$

which is the Volterra equation.

In this case we can enhance the result as [25, Theorem 147].

Theorem 5.11. *If $f(x)e^{-cx} \in L^2(\mathbb{R}^+)$ and $k(x)e^{-cx} \in L^1(\mathbb{R}^+)$ for some $c \in \mathbb{R}$, then there is a unique solution u of*

$$u(x) + \int_0^x k(x-y)u(y)\mathrm{d}y = f(x) \tag{5.57}$$

with $u(x)e^{-c'x} \in L^2(\mathbb{R}^+)$ for large enough c' given by

$$u(x) = \frac{1}{2\pi}\int_{-\infty-ia}^{\infty-ia} \frac{\hat{f}(\omega)}{1+\hat{k}(\omega)} e^{ix\omega}\mathrm{d}\omega \tag{5.58}$$

for large enough a.

Proof. Adding a factor e^{-ax} to u, f, and k does not change the equation, thus by choosing a large enough, it suffices to consider $f \in L^2(\mathbb{R}^+)$, $k \in L^1(\mathbb{R}^+)$, and $\|k\|_1 < 1$.

Hence by the previous theorem there exists a unique solution. Putting back the factor e^{-ax}, the Fourier transformations are shifted by ia and the solution is

$$u(x) = \frac{1}{2\pi}\int_{-\infty-ia}^{\infty-ia} \frac{\hat{f}(\omega)}{1+\hat{k}(\omega)} e^{ix\omega}\mathrm{d}\omega \tag{5.59}$$

where again

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\mathrm{d}x, \tag{5.60}$$

$$\hat{k}(\omega) = \int_{-\infty}^{\infty} k(x)e^{-ix\omega}\mathrm{d}x. \tag{5.61}$$

□

The solution can be written with a resolvent kernel as in [25, Equation 11.5.6 (page 312)].

Theorem 5.12. *If additionally to the previous theorem $k(x)e^{-cx} \in L^2(\mathbb{R}^+)$, then the solution can also be written as*

$$u(x) = f(x) - \int_0^x f(x-y)r(y)\mathrm{d}y \tag{5.62}$$

where r is the resolvent kernel *given by*

$$r(x) + \int_0^x r(x-y)k(y)\mathrm{d}y = k(x). \tag{5.63}$$

Proof. We can split

$$\frac{\hat{f}(\omega)}{1+\hat{k}(\omega)} = \hat{f}(\omega) - \frac{\hat{f}(\omega)\hat{k}(\omega)}{1+\hat{k}(\omega)}. \tag{5.64}$$

Plugging this into the formula for u yields

$$u(x) = f(x) - \int_{-\infty-ia}^{\infty-ia} \hat{f}(\omega)\frac{\hat{k}(\omega)}{1+\hat{k}(\omega)}e^{ix\omega}\mathrm{d}\omega. \tag{5.65}$$

With the additional assumption, we find $\hat{r} := \frac{\hat{k}}{1+\hat{k}} \in L^2$ thus

$$u(x) = f(x) - \int_{-\infty}^{\infty} f(x-y)r(y)\mathrm{d}y \tag{5.66}$$

where r is the inverse Fourier transformation of $\hat{r}$, i.e. with the shift

$$r(x) = \frac{1}{2\pi}\int_{-\infty-ia}^{\infty-ia} \hat{r}(\omega)e^{ix\omega}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty-ia}^{\infty-ia} \frac{\hat{k}(\omega)}{1+\hat{k}(\omega)}e^{ix\omega}\mathrm{d}\omega. \tag{5.67}$$

By considering $a \to \infty$, we find $r(x) = 0$ for $x < 0$, thus the convolution reduces to the given form. The definition $\hat{r} := \frac{\hat{k}}{1+\hat{k}}$ is after Fourier transformation equivalent to eq. (5.63). □

After this motivating treatment following [25] we note from [11, Chapter 2] that we can treat the resolvent kernel more generally. In the rest of this section we will repeat a reduced treatment of [11][5].

For this recall the convolution $f * g$ of two functions f, g defined on $\mathbb{R}$ given for $t \in \mathbb{R}$ as

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)\mathrm{d}s \tag{5.68}$$

whenever the integral exists.

For functions vanishing on $\mathbb{R}^-$ or functions only defined on $\mathbb{R}^+$ the convolution is for $t \in \mathbb{R}^+$

$$(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s \tag{5.69}$$

whenever the integral exists. If we now restrict f and g to be defined on $[0, T]$ for $T > 0$, the convolution is still a meaningful integral for $t \in [0, T]$, i.e. we can define the convolution as function of $[0, T]$ again. Since we take f and g to be scalar valued, by a change of variable $f * g = g * f$. The convolution exists and is bounded by the following theorem.

Theorem 5.13. *Let J be $\mathbb{R}$, $\mathbb{R}^+$ or $[0, T]$ for $T > 0$. If $f \in L^1(J)$ and $g \in L^p(J)$ for $p \in [1, \infty]$, then $f * g \in L^p(J)$ and $\|f * g\|_{L^p(J)} \leq \|f\|_{L^1(J)} \|g\|_{L^p(J)}$.*

Proof. In the case $p \in [1, \infty)$ note that $\mu(ds) = \frac{f(s)}{\|f\|_1}$ defines a probability measure and we can apply Jensen's inequality to find for $t \in J$

$$|(f * g)(t)|^p \leq \left(\int |f(t)|\,|g(t-s)|\,\mathrm{d}s\right)^p \leq \|f\|_1^{p-1} \int |f(s)|\,|g(t-s)|^p\,\mathrm{d}s. \tag{5.70}$$

Integrate over t and use Fubini to find

$$\|f * g\|_p^p \leq \|f\|_1^p \|g\|_p^p. \tag{5.71}$$

In the case $p = \infty$ we can trivially bound

$$|(f * g)(t)| \leq \|f\|_1 \|g\|_\infty. \tag{5.72}$$

□

[5] In this reference more function spaces and matrix-valued functions are considered.

For applications we use the following corollary.

Corollary 5.14. *Let $f \in L^1_{loc}(\mathbb{R}^+)$ and $g \in L^p_{loc}(\mathbb{R}^+)$ for $p \in [1,\infty]$. Then $f * g \in L^p_{loc}$. If $h \in L^1_{loc}(\mathbb{R}^+)$, then also*

$$(f * g) * h = f * (g * h). \tag{5.73}$$

Proof. For $t \in [0,T]$ the convolution $(f * g)(t)$ is already defined by the restriction of f and g to $[0,T]$. Hence by the previous theorem

$$f * g \in L^p_{\mathrm{loc}}. \tag{5.74}$$

This shows that for any $t \in \mathbb{R}$ the integrals of $((f * g) * h)(t)$ and $(f * (g * h))(t)$ are absolutely convergent and we can use Fubini to show associativity. □

We can now introduce the resolvent kernel more generally.

Theorem 5.15. *Let $k \in L^1_{loc}(\mathbb{R}^+)$. Then there exists a solution $r \in L^1_{loc}(\mathbb{R}^+)$ of*

$$r + r * k = k \tag{5.75}$$

which is unique. This solution is called resolvent kernel.

For the proof we use the following lemma to reduce the considered norm.

Lemma 5.16. *Let $r \in L^1_{loc}(\mathbb{R}^+)$ be the resolvent kernel of $k \in L^1_{loc}(\mathbb{R}^+)$, then for any $\sigma \in \mathbb{C}$ the function $e^{\sigma t} r(t) \in L^1_{loc}(\mathbb{R}^+)$ is the resolvent kernel of $e^{\sigma t} k(t) \in L^1_{loc}(\mathbb{R}^+)$.*

Proof. For any $t \in \mathbb{R}$ multiply $r + r * k = k$ by $e^{\sigma t}$, then

$$r(t)e^{\sigma t} + \int_0^t \left(r(t-s)e^{\sigma(t-s)}\right)(g(s)e^{\sigma s})\,\mathrm{d}s = k(t)e^{\sigma t} \tag{5.76}$$

which is the claim. □

Proof of theorem 5.15. First we show uniqueness. Suppose $r, \bar{r} \in L^1_{\mathrm{loc}}(\mathbb{R}+)$ are both solutions. Then by the shown associativity

$$\bar{r} * k = \bar{r} * (r + r * k) = r * (\bar{r} + \bar{r} * k) = r * k \tag{5.77}$$

and so $r = k - r * k = k - \bar{r} * k = \bar{r}$.

With the uniqueness we can now show that it suffices to construct a solution r_T on $[0,T]$ satisfying for $t \in [0,T]$

$$r(t) + (t * k)(t) = k(t). \tag{5.78}$$

If this is true we can construct solutions r_j on $[0,j]$ for $j \in \mathbb{N}$ and the restriction of r_k for $k > j$ to $[0,j]$ must equal r_j by the uniqueness. Hence we can define $r \in L^1_{\mathrm{loc}}$ by $r(t) = r_j(t)$ for $t \in [j-1,j)$ which satisfies for all $t \in \mathbb{R}$ the required relation $r(t) + (r * k)(t) = k(t)$.

Hence we can restrict to $[0,T]$. Now by dominated convergence

$$\int_0^T \left|k(t)e^{-ct}\right|\mathrm{d}t \tag{5.79}$$

converges to 0 as $c \to \infty$, so by lemma 5.16 we can assume $\int_0^T |k(t)|\,\mathrm{d}t < 1$. Let k^n be the n-times convolution of k with itself, i.e.

$$k^{*1} = k, \qquad k^{*n} = k^{*(n-1)} * k. \tag{5.80}$$

By theorem 5.13, $\|k^{*n}\|_{L^1([0,T])} \leq \|k\|^n_{L^1([0,T])}$. Hence

$$r_m = \sum_{j=1}^m (-1)^{j-1} k^{*j} \tag{5.81}$$

is a Cauchy sequence in $L^1([0,T])$ and by completeness it converges to $r \in L^1([0,T])$, say. From the definition we see for all $m \geq 1$

$$r_m + r_{m-1} * k = k. \tag{5.82}$$

Since by theorem 5.13 also $r_{m-1} * k \to r * k$ as $m \to \infty$, taking the limit gives

$$r + r * k = k. \tag{5.83}$$

□

With the resolvent kernel we can solve the Volterra equation.

Theorem 5.17. *Let $k \in L^1_{loc}(\mathbb{R}^+)$. Then for every $f \in L^1_{loc}(\mathbb{R}^+)$ there exists a unique solution $u \in L^1_{loc}$ of*

$$u + k * u = f \tag{5.84}$$

given by

$$u = f - f * r \tag{5.85}$$

where $r \in L^1_{loc}$ is the resolvent kernel of k.

Proof. By the previous theorem there exists a resolvent kernel $r \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ satisfying $r + r*k = k$. If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ is a solution then

$$r * f = r * (u + k * u) = u * (r + r * k) = u * k \tag{5.86}$$

and thus

$$u = f - f * r \tag{5.87}$$

which proves uniqueness.

On the other hand if $u = f - f * r$ then

$$u + k * u = (f - f * r) + k * (f - f * r) = f + f * (k - r) - f * (k * r) = f \tag{5.88}$$

so this u is a solution. □

For the long term behaviour of the solution we want to characterise the integrability of the resolvent kernel. In the case of an integrable kernel this can be answered completely by the Paley-Wiener theorem (original reference [19, Chapter 18], our presentation will continue to follow [11, Chapter 2]).

Theorem 5.18 (Paley-Wiener)**.** *Let $k \in L^1(\mathbb{R}^+)$. The resolvent kernel $r \in L^1_{loc}(\mathbb{R}^+)$ satisfying*

$$r + r * k = k \tag{5.89}$$

is in $L^1(\mathbb{R}^+)$ iff for all $p \in \mathbb{C}$ with $\Re(p) \geq 0$

$$\tilde{k}(p) := \int_0^\infty k(t) e^{-pt} \mathrm{d}t \neq -1. \tag{5.90}$$

The condition is necessary as we can see directly by Laplace transformation. If $k, r \in L^1$ then their Laplace transformations $\tilde{k}, \tilde{r}$ are continuous and bounded in the right half plane $\Re(p) \geq 0$. Furthermore they satisfy for $p \in \mathbb{C}$ with $\Re(p) \geq 0$

$$\tilde{r}(p) + \tilde{r}(p)\tilde{k}(p) = \tilde{k}(p) \quad \Rightarrow \quad \tilde{r}(p)[1 + \tilde{k}(p)] = \tilde{k}(p) \tag{5.91}$$

which can only be true if $\tilde{k}(p) \neq -1$.

For proving the sufficiency, the usage of the assumption becomes more apparent if we follow [11] instead of the original work [19] which uses the following theorem as intermediate step.

Theorem 5.19 (Whole Line Paley-Wiener)**.** *Let $k \in L^1(\mathbb{R})$. There exists a function $r \in L^1(\mathbb{R})$ satisfying $r + r * k = k$ iff for all $p \in \mathbb{C}$ with $\Re(p) = 0$*

$$\tilde{k}(p) := \int_{-\infty}^{\infty} k(t)e^{-pt}\mathrm{d}t \neq -1. \tag{5.92}$$

Here we extended the Laplace transformation to the (bilateral) Laplace transformation. For $p = i\omega$ with $\omega \in \mathbb{R}$, then $\tilde{k}(i\omega)$ becomes the Fourier transformation which is bounded. Hence by the same argument the condition is necessary.

Using this theorem we can prove theorem 5.18.

Proof of theorem 5.18 using theorem 5.19. We are only left to show the sufficiency of the criterion.

Suppose $k \in L^1(\mathbb{R}^+)$ with $\tilde{k}(p) \neq -1$ for all $p \in \mathbb{C}$ with $\Re(p) \geq 0$. Then we extend k to a function in $L^1(\mathbb{R})$ with $k(x) = 0$ for $x < 0$. Then still $\tilde{k}(p) \neq -1$ for all $p \in \mathbb{C}$ with $\Re(p) \geq 0$.

By theorem 5.19 there exists $r \in L^1(\mathbb{R})$ with $r + r * k = k$ and we split r into r_- and r_+ as

$$r_-(t) = \begin{cases} r(t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases} \qquad r_+(t) = \begin{cases} 0 & \text{if } t < 0 \\ r(t) & \text{if } t \geq 0 \end{cases} \tag{5.93}$$

so that we are only left to show that r_- is vanishing since then we can restrict r to a function of $\mathbb{R}^+$ and recover the half-line convolution.

For $p \in \mathbb{C}$ with $\Re(p) = 0$ we have as $\tilde{k}(p) \neq -1$ the relation

$$\tilde{r}(p) = \tilde{k}(p)[1 + \tilde{k}(p)]^{-1}. \tag{5.94}$$

By linearity this is

$$\tilde{r}_-(p) = \frac{\tilde{k}(p)}{1 + \tilde{k}(p)} - \tilde{r}_+(p). \tag{5.95}$$

The LHS now is an analytic function of p for $\Re(p) < 0$ which is continuous and bounded for $\Re(p) \leq 0$. Since $\tilde{k}(p) \neq -1$ for $\Re(p) \geq 0$, the RHS is an analytic function of p for $\Re(p) > 0$. As $\tilde{k}(p) \to 0$ uniformly as $|p| \to \infty$ for $\Re(p) \geq 0$, by continuity there exists a $\delta > 0$ such that $\left|1 + \tilde{k}(p)\right| > \delta$ for all $\Re(p) \geq 0$. Hence the RHS is also continuous and bounded for $\Re(p) \geq 0$. Therefore the LHS and RHS must be part of the same bounded entire function which must be constant. As $\tilde{r}_-(p) \to 0$ as $p \to -\infty$, this constant is 0 and so by the uniqueness of the Fourier transformation r_- is vanishing. □

The proof of theorem 5.19 is more conveniently written in terms of the Fourier transformation. For $k, r \in L^1(\mathbb{R})$ introduce for $\omega \in \mathbb{R}$ the Fourier transformations

$$\hat{k}(\omega) = \int k(x)e^{-ix\omega}\mathrm{d}x \tag{5.96}$$

$$\hat{r}(\omega) = \int r(x)e^{-ix\omega}\mathrm{d}x \tag{5.97}$$

and the condition $r + r * k = k$ for the resolvent kernel is

$$\hat{r}(\omega) + \hat{r}(\omega)\hat{k}(\omega) = \hat{k}(\omega). \tag{5.98}$$

On the other hand if we can find $r \in L^1(\mathbb{R})$ such that for all ω its Fourier transformation satisfies $\hat{r}(\omega) + \hat{r}(\omega)\hat{k}(\omega) = \hat{k}(\omega)$ then by the uniqueness of the Fourier transformation $r + r * k - k$, as then $r + r * k - k$ has vanishing Fourier transformation and must be vanishing by theorem 5.3.

Hence we are going to prove that such a $r \in L^1(\mathbb{R})$ exists. For this let $\hat{f}$ denote the Fourier transformation of a function f. We are going to do this like in the proof of theorem 5.15 by a series argument and split ω in ranges to control the norm. With a restriction on the norm, we can construct a building block by the following lemma.

Lemma 5.20. *Let $f, g \in L^1(\mathbb{R})$ and $\|g\|_1 < 1$. Then there exists $h \in L^1(\mathbb{R})$ such that for all $\omega \in \mathbb{R}$*

$$\hat{h}(\omega) = \frac{\hat{f}(\omega)}{1 + \hat{g}(\omega)}. \tag{5.99}$$

Proof. As $|\hat{g}(\omega)| \leq \|g\|_1 < 1$ for all $\omega \in \mathbb{R}$ the denominator is well-defined. Again denote with g^* the n-times convolution of g with itself, i.e. $g^{*1} = g$ and $g^{*n} = g^{*(n-1)} * g$.

Then by theorem 5.13 $\|g^{*n}\|_1 \leq \|g\|_1^n$ so

$$h_n = f * \sum_{i=0}^{n} (-1)^i g^{*i} \tag{5.100}$$

is a Cauchy sequence which converges to $h \in L^1(\mathbb{R})$. Also for $n \geq 1$

$$h_n + g * h_{n-1} = f \tag{5.101}$$

and taking $n \to \infty$, the terms converge separately to

$$h + g * h = f. \tag{5.102}$$

Hence by taking the Fourier transformation

$$\hat{h}(\omega) = \frac{\hat{f}(\omega)}{1 + \hat{g}(\omega)} \tag{5.103}$$

for all $\omega \in \mathbb{R}$. □

In order to localise the problem in Fourier space we introduce the Fejér kernel

$$\zeta(t) = \frac{1}{\pi t^2}(1 - \cos t) \tag{5.104}$$

with Fourier transformation

$$\hat{\zeta}(\omega) = \begin{cases} 1 - |\omega| & \text{if } \omega \leq 1 \\ 0 & \text{if } \omega > 1 \end{cases} \tag{5.105}$$

Further define

$$\eta(t) = 4\zeta(2t) - \zeta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t) \tag{5.106}$$

with Fourier transformation

$$\hat{\eta}(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq 1 \\ 2 - |\omega| & \text{if } 1 < |\omega| \leq 2 \\ 0 & \text{if } |\omega| > 2 \end{cases} \tag{5.107}$$

which is a localised function constant around the origin. Finally, we can scale these by δ as

$$\zeta_\delta(t) = \delta\zeta(\delta t), \quad \eta_\delta(t) = \delta\eta(\delta t) \tag{5.108}$$

so that

$$\hat{\zeta}_\delta(\omega) = \zeta\left(\frac{\omega}{\delta}\right), \quad \hat{\eta}_\delta(\omega) = \eta\left(\frac{\omega}{\delta}\right). \tag{5.109}$$

With this we can prove the following lemma which allows us to localise the Fourier transformation around a point.

Lemma 5.21. *Let $f \in L^1(\mathbb{R})$, $\epsilon > 0$, $\omega_0 \in \mathbb{R}$. Then there exists a number δ independent of ω_0 and $g \in L^1(\mathbb{R})$ with $\|g\|_1 \leq \epsilon$ satisfying*

$$\hat{f}(\omega) = \hat{f}(\omega_0) + \hat{g}(\omega) \tag{5.110}$$

for $|\omega - \omega_0| \leq \delta$.

Proof. For $\delta < 0$ and η_δ from before consider

$$\hat{g}_\delta(\omega) = \hat{\eta}_\delta(\omega - \omega_0)(\hat{f}(\omega) - \hat{f}(\omega_0)). \tag{5.111}$$

For $|\omega - \omega_0| \leq \delta$ we have $\hat{\eta}_\delta(\omega - \omega_0) = 1$ and so $\hat{g}_\delta(\omega) = \hat{f}(\omega) - \hat{f}(\omega_0)$. Further $\hat{g}_\delta$ is the Fourier transformation of g_δ given by

$$\begin{aligned} g_\delta(t) &= \int_{\mathbb{R}} e^{i\omega_0(t-s)} \eta_\delta(t-s) f(s) \mathrm{d}s - e^{i\omega_0 t} \eta_\delta(t) \int_{\mathbb{R}} f(s) e^{-i\omega_0 s} \mathrm{d}s \\ &= \int_{\mathbb{R}} e^{i\omega_0(t-s)} [\eta_\delta(t-s) - \eta_\delta(t)] f(s) \mathrm{d}s. \end{aligned} \tag{5.112}$$

Hence using Fubini on the absolute value

$$\|g_\delta\|_1 \leq \int_{\mathbb{R}} |f(s)| \int_{\mathbb{R}} |\eta_\delta(t-s) - \eta_\delta(t)| \, \mathrm{d}t \mathrm{d}s \tag{5.113}$$

which is an estimate independent of ω_0. Furthermore the integrand is bounded by

$$|f(s)| \int_{\mathbb{R}} |\eta_\delta(t-s) - \eta_\delta(t)| \, \mathrm{d}t \leq 2 \, |f(s)| \, \|\eta_\delta\|_1 \tag{5.114}$$

and pointwise converging to 0 as $\delta \to 0$. Hence by dominated convergence the bound tends to 0 as $\delta \to 0$. Thus we can make $\|g_\delta\|_1$ independent of ω_0 as small as we want. □

The large frequencies can be handled by the following lemma.

Lemma 5.22. *Let $f \in L^1(\mathbb{R})$ and $\epsilon > 0$. There exists $M > 0$ and $g \in L^1(\mathbb{R})$ with $\|g\|_1 \leq \epsilon$ satisfying*

$$\hat{f}(\omega) = \hat{g}(\omega) \tag{5.115}$$

for $|\omega| \geq M$.

Proof. Use the Fejér kernel ζ_ρ and consider

$$\hat{g}(\omega) = \hat{f}(\omega) - \hat{\zeta}_\rho(\omega) \hat{f}(\omega). \tag{5.116}$$

Since $\hat{\zeta}_\rho$ is vanishing for $|\omega| \geq \rho$, we have for $|\omega| \geq \rho$

$$\hat{g}(\omega) = \hat{f}(\omega). \tag{5.117}$$

Further $\hat{g}$ is the Fourier transformation of g given by

$$\begin{aligned} g(t) &= f(t) - \int \zeta_\delta(s) f(t-s) \mathrm{d}s \\ &= f(t) - \int \zeta(s) f(t - \frac{s}{\delta}) \mathrm{d}s \\ &= \int \zeta(s)(f(t) - f(t - \frac{s}{\delta})) \mathrm{d}s \end{aligned} \tag{5.118}$$

where we used a change of variables and that $\int \zeta(s) \mathrm{d}s = 1$ which can straightforward be calculated by contour integration. Hence using Fubini we find for the absolute value

$$\|g\|_1 \leq \int \zeta(s) \left\| f - f_{s/\delta} \right\|_1 \mathrm{d}s \tag{5.119}$$

where f_s is the shifted function, i.e. $f_s(x) = f(x-s)$. By dominated convergence $\|f - f_s\|_1 \to 0$ as $s \to 0$. Since $\|f - f_s\|_1 \leq 2 \|f\|_1$, by dominated convergence again, $\|g\|_1 \to 0$ as $\delta \to \infty$. □

Now we can put the pieces together to prove the remaining sufficiency of the whole line Paley-Wiener theorem.

Proof of theorem 5.19. From the remarks before it remains to prove that for $k \in L^1(\mathbb{R})$ with $\hat{k}(\omega) \neq -1$ for all $\omega \in \mathbb{R}$ there exists $r \in L^1(\mathbb{R})$ with $\hat{r}(\omega)+\hat{r}(\omega)\hat{k}(\omega) = \hat{k}(\omega)$, where we expressed the condition in terms of the Fourier transformation instead of the Laplace transformation.

By the Riemann-Lebesque lemma $\hat{k}(\omega) \to 0$ as $\omega \to \pm\infty$. Hence by continuity there exists $A > 0$ with $\left|1 + \hat{k}(\omega)\right| \geq A$ for all $\omega \in \mathbb{R}$.

By lemma 5.22 there exists M and $g_\infty \in L^1(\mathbb{R})$ with $\|g_\infty\|_1 < 1$ and $\hat{k}(\omega) = \hat{g}_\infty(\omega)$ for $|\omega| \geq M$. Whithout loss of generality we can take $M \in \mathbb{N}$.

Next by lemma 5.21 we can find $m \in \mathbb{N}$ and $g_j \in L^1(\mathbb{R})$ for $j = -mM, \ldots, -1, 0, 1, \ldots, mM$ with $\|g_j\|_1 < A$ and $\hat{k}(\omega) = \hat{k}(\frac{j}{m}) + \hat{g}_j(\omega)$ for $\left|\omega - \frac{j}{m}\right| \leq \frac{1}{m}$. Then for $|\omega - \omega_0| \leq \frac{1}{m}$ where $\omega_0 = \frac{j}{m}$

$$\frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)} = \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega_0)} \left[1 + \frac{\hat{g}_j(\omega)}{1 + \hat{k}(\omega_0)}\right]^{-1} \tag{5.120}$$

where $\left\|\frac{g_j(\omega)}{1+\hat{k}(\omega_0)}\right\|_1 \leq \frac{\|g_j\|_1}{A} < 1$ and $1 + \hat{k}(\omega_0)$ is just a number. Hence by applying lemma 5.20 to g_∞ respectively $\frac{g_j}{1+\hat{k}(\omega_0)}$ there exist $r_\infty \in L_1$ and $r_j \in L_1$ for $j = -mM, \ldots, mM$ such that

$$\hat{r}_\infty(\omega) = \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)} \quad \text{for } |\omega| \geq M \tag{5.121}$$

$$\hat{r}_j(\omega) = \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)} \quad \text{for } \left|\omega - \frac{j}{m}\right| \leq \frac{1}{m}. \tag{5.122}$$

Finally define the shifted Fejér kernel $\psi_j(t) = \frac{1}{m} e^{-ijt/m} \zeta(\frac{t}{m})$ with Fourier transformation

$$\hat{\psi}_j(\omega) = \begin{cases} 1 - |m\omega - j|, & \left|\omega - \frac{j}{m}\right| \leq \frac{1}{m} \\ 0, & \left|\omega - \frac{j}{m}\right| > 1 \end{cases} \tag{5.123}$$

and let

$$r = \sum_{j=-mM}^{mM} \psi_j * (r_j - r_\infty) + r_\infty \tag{5.124}$$

which is integrable as it is a finite sum of integrable functions. By the shape of the Fejér kernel for all $|\omega| \leq M$

$$\sum_{j=-mM}^{mM} \hat{\psi}_0(\omega) = 1. \tag{5.125}$$

Thus for all $\omega \in \mathbb{R}$

$$\left(1 - \sum_{j=-mM}^{mM} \hat{\psi}_j(\omega)\right) \hat{r}_\infty(\omega) = \left(1 - \sum_{j=-mM}^{mM} \hat{\psi}_j(\omega)\right) \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)}. \tag{5.126}$$

On the other hand $\hat{\psi}_j$ has support $\left|\omega - \frac{\delta}{m}\right| \leq \frac{1}{m}$ and thus for all $\omega \in \mathbb{R}$

$$\hat{\psi}_j(\omega)\hat{r}_j(\omega) = \hat{\psi}_j(\omega) \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)}. \tag{5.127}$$

Adding this for $j = -mM, \ldots, mM$ proves for all $\omega \in \mathbb{R}$ the desired relation

$$\hat{r}(\omega) = \frac{\hat{k}(\omega)}{1 + \hat{k}(\omega)}. \tag{5.128}$$

□

5.5 Principal Value and Plemelj Formula

In our analysis we considered the limit of a Cauchy type integral. This limit can be calculated by the Plemelj formula which we prove here.

The Plemelj formula states for distributions $D'(\mathbb{R})$ of $\mathbb{R}$

$$\frac{1}{x - i0} = \mathrm{PV}\left(\frac{1}{x}\right) + i\pi\delta_0 \tag{5.129}$$

or equivalently

$$\frac{1}{x + i0} = \mathrm{PV}\left(\frac{1}{x}\right) - i\pi\delta_0. \tag{5.130}$$

The principle value PV is the prescription to cancel the divergence by symmetry.

Definition 5.23 (Principle Value)**.** *For an interval $[a, b] \in \mathbb{R}$ let $f : [a, b] \to \mathbb{C}$ be a function which diverges at $c \in (a, b)$. Then define*

$$\mathrm{PV}\int_a^b f(x)\mathrm{d}x = \lim_{\epsilon\to 0}\left(\int_a^{c-\epsilon} + \int_{c+\epsilon}^b\right) f(x)\mathrm{d}x. \tag{5.131}$$

For $f : \mathbb{R} \to \mathbb{C}$ define

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda} f(x)\mathrm{d}x. \tag{5.132}$$

These integrals with principle value are also sometimes called *singular integral* [16].

From [17] we adapt a stronger and more concrete form which we enhance to cover convergence from the complex plane as suggested by [18, Chapter 2][6].

Theorem 5.24. *If $f \in L^1$ is a Lipschitz continuous function, then*

$$\lim_{\lambda\to 0+0}\int_{\mathbb{R}}\frac{f(x)}{x - i\lambda}\mathrm{d}x = \mathrm{PV}\int\frac{f(x)}{x}\mathrm{d}x + i\pi f(0). \tag{5.133}$$

Moreover, the rate of convergence can be bounded by the Lipschitz constant K, a L^1 norm bound M_1, and a L^∞ norm bound M_∞, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ only depending on K, M_1, and M_∞ such that

$$\left|\int_{\mathbb{R}}\frac{f(x)}{x - i\lambda}\mathrm{d}x - \left(\mathrm{PV}\int\frac{f(x)}{x}\mathrm{d}x + i\pi f(0)\right)\right| \le \epsilon \tag{5.134}$$

holds for all $\lambda \in (0, \delta)$ and for all $f \in L^1$ satisfying the norm bounds $\|f\|_1 \le M_1$ and $\|f\|_\infty \le M_\infty$ and the Lipschitz continuity $|f(x) - f(y)| \le K\,|x - y|$ for all $x, y \in \mathbb{R}$.

Proof. Introduce an auxiliary Lipschitz continuous function $g \in L^1$ which is even (i.e. $g(x) = g(-x)$ for all $x \in \mathbb{R}$) and $g(0) = 1$.

Since f and g are Lipschitz continuous, $\frac{f(x)-f(0)}{x}$ and $\frac{1-g(x)}{x}$ are bounded. Hence

$$\begin{aligned}
\mathrm{PV}\int\frac{f(x)}{x}\mathrm{d}x &= \lim_{\epsilon\to 0}\left(\int_{|x|\ge\epsilon}\frac{f(x)}{x}g(x)\mathrm{d}x + \int_{|x|\ge\epsilon} f(x)\frac{1 - g(x)}{x}\mathrm{d}x\right)\\
&= \lim_{\epsilon\to 0}\left(\int_{|x|\ge\epsilon}\frac{f(x) - f(0)}{x}g(x)\mathrm{d}x + \int_{|x|\ge\epsilon} f(x)\frac{1 - g(x)}{x}\mathrm{d}x\right)\\
&= \int_{\mathbb{R}}\frac{f(x) - f(0)}{x}g(x)\mathrm{d}x + \int_{\mathbb{R}} f(x)\frac{1 - g(x)}{x}\mathrm{d}x
\end{aligned} \tag{5.135}$$

where we used that $\int_{|x|\ge\epsilon} f(0)\frac{g(x)}{x}\mathrm{d}x = 0$ as g is even.

[6]His proof uses $g(x) = 1$ and is generalised to any contour.

On the other hand for $\lambda > 0$ the integral $\int_{\mathbb{R}} \frac{f(x)}{x-i\lambda}\mathrm{d}x$ is convergent and we can split it as

$$\int_{\mathbb{R}} \frac{f(x)}{x-i\lambda}\mathrm{d}x = \int_{\mathbb{R}} \frac{f(x)-f(0)}{x-i\lambda} g(x)\mathrm{d}x + \int_{\mathbb{R}} f(x)\frac{1-g(x)}{x-i\lambda}\mathrm{d}x + f(0)\int_{\mathbb{R}} \frac{g(x)}{x-i\lambda}\mathrm{d}x. \tag{5.136}$$

Since f and g are Lipschitz continuous, the first two integrals converge by dominated convergence as $\lambda \to 0+0$ to the RHS of eq. (5.135), i.e. to PV $\int \frac{f(x)}{x}\mathrm{d}x$.

Therefore, we are left to show that $\lim_{\lambda\to 0+0}\int_{\mathbb{R}} \frac{g(x)}{x-i\lambda}\mathrm{d}x = i\pi$. For this we may choose g as $g(x) = e^{-x^2}$ which we can extend to an analytic function of $\mathbb{C}$. Hence we can integrate equally along a contour D_ϵ intended at 0 with an arc of radius ϵ as shown in fig. 9.

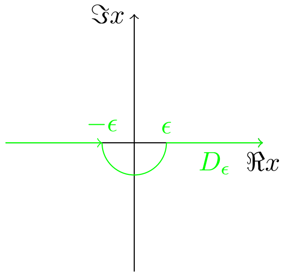

Figure 9 – Contour D_ϵ in proving Plemelj formula

Then $\int_{D_\epsilon} \frac{g(x)}{x-i\lambda}\mathrm{d}x$ converges to $\int_{D_\epsilon} \frac{g(x)}{x}\mathrm{d}x$. Finally, the contributions along the real axis in $\int_{D_\epsilon} \frac{g(x)}{x}\mathrm{d}x$ cancel by symmetry and we are only left with the arc which is as $\epsilon \to 0$ equal to $i\pi$ by the indentation lemma or directly the limit is $\int_{\theta=\pi}^{2\pi} \frac{e^{-2\pi i\theta}}{\epsilon} i\epsilon e^{2\pi i\theta} = i\pi$.

With this g we can also show the explicit bound given the positive constants K, M_1, M_∞, and ϵ. We can choose δ small enough such that for all $\lambda \in (0, \delta)$

$$\left| \int_{\mathbb{R}} \frac{g(x)}{x-i\lambda}\mathrm{d}x - i\pi \right| \leq \frac{\epsilon}{3M_\infty} \tag{5.137}$$

and so as $\|f\|_\infty \leq M_\infty$

$$\left| f(0)\int_{\mathbb{R}} \frac{g(x)}{x-i\lambda}\mathrm{d}x - i\pi f(0) \right| \leq \frac{\epsilon}{3}. \tag{5.138}$$

Next let $r = \min\left(\frac{\epsilon}{24M_\infty}, \frac{\epsilon}{24K}\right)$ then

$$\left| \int_{|x|\leq r} \frac{f(x)-f(0)}{x-i\lambda} g(x)\mathrm{d}x - \int_{|x|\leq r} \frac{f(x)-f(0)}{x} g(x)\mathrm{d}x \right| \leq \frac{\epsilon}{6} \tag{5.139}$$

since $r \leq \frac{\epsilon}{24K}$ and $\left|\frac{f(x)-f(0)}{x-i\lambda} g(x)\right| \leq K$ for all x and λ (including 0). Similarly

$$\left| \int_{|x|\leq r} f(x)\frac{1-g(x)}{x-i\lambda}\mathrm{d}x - \int_{|x|\leq r} f(x)\frac{1-g(x)}{x}\mathrm{d}x \right| \leq \frac{\epsilon}{6} \tag{5.140}$$

since $r \leq \frac{\epsilon}{24M_\infty}$ and $\left|f(x)\frac{1-g(x)}{x-i\lambda}\right| \leq M_\infty$ for all x and λ.

For $|x| \geq r$ on the other hand $\frac{1}{x-i\lambda}$ converges uniformly to $\frac{1}{x}$, so we can choose δ small enough such that for all $\lambda \in (0, \delta)$

$$\left| \int_{|x|>r} \frac{f(x)-f(0)}{x-i\lambda} g(x)\mathrm{d}x - \int_{|x|>r} \frac{f(x)-f(0)}{x} g(x)\mathrm{d}x \right| \leq \frac{\epsilon}{6} \tag{5.141}$$

and

$$\left| \int_{|x|>r} f(x) \frac{1-g(x)}{x-i\lambda} \mathrm{d}x - \int_{|x|>r} f(x) \frac{1-g(x)}{x} \mathrm{d}x \right| \leq \frac{\epsilon}{6} \tag{5.142}$$

using that $\|f\|_1 < M_1$.

By the triangle inequality we can put everything together which shows the explicit bound. □

From the Lipschitz condition we could conclude $\|f\|_1 \geq \|f\|_\infty K$ as we can see in fig. 10 so that we could drop the L^∞ bound in the theorem.

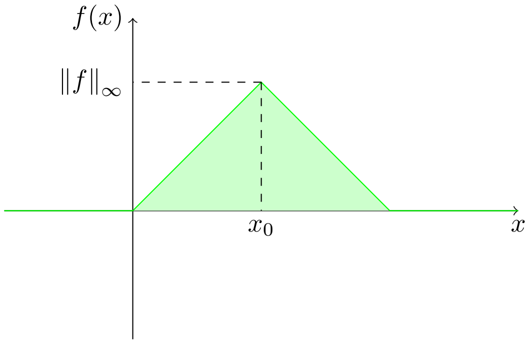

Figure 10 – Illustration why $\|f\|_1 \geq K \|f\|_\infty$ for a Lipschitz continuous function f with maximal absolute value at x_0.

For our application the following corollary is important.

Corollary 5.25. *If $f \in L^1$ is a Lipschitz continuous function, then*

$$\int_{\mathbb{R}} \frac{f(x+y)}{x-i\lambda} \mathrm{d}x \to \mathrm{PV} \int \frac{f(x+y)}{x} \mathrm{d}x + i\pi f(y) \tag{5.143}$$

uniformly over $y \in \mathbb{R}$ as $\lambda \to 0+0$.

Proof. Use the general bound from the previous theorem as the shifted function $f_y(x) = f(y+x)$ satisfies the same Lipschitz condition and has the same bounds. □

Since for $\lambda > 0$, the integral $\int_{\mathbb{R}} \frac{f(x+y)}{x-i\lambda} \mathrm{d}x$ is a continuous function of y, the uniform convergence shows that the limit is continuous.

Finally we remark that the Lipschitz continuity can be replaced by Hölder continuity as done in [18].

6 Outlook

The great achievement of Mouhot and Villani [17] after more than 60 years was to show that Landau damping follows from the full Vlasov equation, which we did not discuss at all in this essay. We also did not look further how the full distribution evolves under damping which even in the linearized case can lead to non-trivial effects like the plasma echo [10]. A future research could further analyse this behaviour under the full Vlasov equation. Another view on linear Landau damping are the Van Kampen modes which are more difficult to justify. A comparison by Backus can be found in [1].

Mathematically, the analysis can be expressed by semi-groups and spectral analysis of the involved operators. This gives a very powerful framework to formulate and understand the results. Also it may be enlightening to construct a solution f_1 with initial integrable but not square integrable solution and unbounded $\int \left| \int f_1(t, v_x, v_y, v_z) \mathrm{d}v_y \mathrm{d}v_z \right| \mathrm{d}v_x$ as proposed by Backus explicitly to compare it with the Landau damping.

Physically, for a collision free plasma there are interesting effects due to magnetic fields. A development of the corresponding physical equations can be found in [23]. Since the magnetic field couples to the velocity, we cannot simply change the interaction potential. An example where magnetic fields are crucial is the corona of the sun or in attempts to build a fusion reactor where the plasma is confined with a magnetic field.

A Notational Overview

Throughout the essay we will use the notation $\frac{\partial f(x,y)}{\partial x}$ for a partial derivative with respect to the first variable at the point (x, y).

Let e, m be the charge respectively the mass of a particle. If we have different kind of particles, we use greek indices.

The distribution of particles is given by $f(t, \mathbf{x}, \mathbf{v})$ where $f(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$ is the number of particles with position in $[\mathbf{x}, \mathbf{x} + \mathrm{d}\mathbf{x}]$ and velocity $[\mathbf{v}, \mathbf{v} + \mathrm{d}\mathbf{v}]$. The evolution is determined by the Vlasov equation

$$\frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{A.1}$$

$$\nabla^2 \phi(t,\mathbf{x}) = -4\pi \left(\rho_b + e \int f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v} \right) \tag{A.2}$$

where $\rho_b(\mathbf{x})$ is the distribution of background charges and the electric potential ϕ can be replaced by the electric field $\mathbf{E} = -\nabla\phi$. Furthermore, the differential equation for ϕ can be replaced using the fundamental solution by

$$\phi(t,\mathbf{x}) = \int \left(\frac{e}{|\mathbf{x}-\mathbf{y}|} \int f(t,\mathbf{y},\mathbf{v})\mathrm{d}\mathbf{v} \right) \mathrm{d}\mathbf{y}. \tag{A.3}$$

For several kind of particles we find

$$\frac{\partial f_\alpha(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e_\alpha}{m_\alpha}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f_\alpha(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} = 0 \tag{A.4}$$

$$\nabla^2 \phi(t,\mathbf{x}) = -4\pi \left(\rho_b + \sum_\alpha e_\alpha \int f_\alpha(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v} \right). \tag{A.5}$$

Consider a spatial homogenous distribution $f_0(\mathbf{v})$ and background charges $\rho_b = e\int f_0(\mathbf{v})\mathrm{d}\mathbf{v}$. This is a solution and we consider a small perturbation $f_1(t, \mathbf{x}, \mathbf{v})$. The linearised equation for $f_1(t, \mathbf{x}, \mathbf{v})$ is

$$\frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{A.6}$$

$$\nabla^2 \phi(t,\mathbf{x}) = -4\pi e \int f_1(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}. \tag{A.7}$$

For several kinds of particles

$$\frac{\partial f_{1\alpha}(t,\mathbf{x},\mathbf{v})}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{1\alpha}(t,\mathbf{x},\mathbf{v})}{\partial \mathbf{x}} - \frac{e}{m}\frac{\partial \phi(t,\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f_{0\alpha}(\mathbf{v})}{\partial \mathbf{v}} = 0, \tag{A.8}$$

$$\nabla^2 \phi(t,\mathbf{x}) = \sum_\alpha -4\pi e_\alpha \int f_{1\alpha}(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}. \tag{A.9}$$

By linearity consider spatial Fourier mode $\mathbf{k}$ separately which evolves by

$$\frac{\partial f_1(t,\mathbf{v})}{\partial t} + ikv_x f_1(t,\mathbf{v}) - i\phi(t)\frac{ke}{m}\frac{\partial f_0(\mathbf{v})}{\partial v_x} = 0, \tag{A.10}$$

$$k^2\phi(t) = 4\pi e \int f_1(t, \mathbf{v})\mathrm{d}\mathbf{v}. \tag{A.11}$$

where we have chosen $\mathbf{k}$ along the x-axis.

A growing mode with frequency ω ($\Im(\omega) > 0$) exists if

$$k^2 = Z\left(\frac{\omega}{k}\right) \tag{A.12}$$

where

$$Z(s) = \int \frac{\mathrm{d}h(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{u-s}, \tag{A.13}$$

$$h(u) = \omega_p^2 g_0(u), \tag{A.14}$$

$$g_0(u) = \frac{1}{n}\int f_0(u, v_y, v_z)\mathrm{d}v_y\mathrm{d}v_z \tag{A.15}$$

and the (unperturbed) plasma frequency $\omega_p = \sqrt{4\pi n e^2/m}$ absorbes the constants with the (unperturbed) density $n = \int f_0(\mathbf{v})\mathrm{d}\mathbf{v}$. For several kinds of particles change

$$h(u) = \sum_\alpha \omega_{p\alpha}^2 g_{0\alpha}(u) \tag{A.16}$$

where

$$g_{0\alpha}(u) = \frac{1}{n_\alpha}\int f_{0\alpha}(u, v_y, v_z)\mathrm{d}v_y\mathrm{d}v_z. \tag{A.17}$$

References

[1] George Backus. Linearized plasma oscillations in arbitrary electron velocity distributions. *Journal of Mathematical Physics*, 1(3):178–191, 1960.

[2] Lothar Berg. *Einführung in die Operatorenrechnung.* Deutscher Verlag der Wissenschaften, Berlin, 1962.

[3] Ira B. Bernstein. Waves in a plasma in a magnetic field. *Phys. Rev.*, 109:10–21, Jan 1958.

[4] Ludwig Boltzmann. Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Wiener Berichte*, 66:275–370, 1872. Reproduced in *Entropie und Wahrscheinlichkeit*, edited and with an introduction by Dieter Flamm, Verlag Harri Deutsch, 2000.

[5] A.R. Choudhuri. *The physics of fluids and plasmas: an introduction for astrophysicists.* Cambridge University Press, 1998.

[6] Gustav Doetsch. *Einführung in Theorie und Anwendung der Laplace-Transformation; ein Lehrbuch für Studierende der Mathematik, Physik und Ingenieurwissenschaft.* Birkhäuser, Basel, 1958.

[7] Dietrich Dreyer. Ein Plädoyer für den Einsatz des Operatorenkalküls nach Mikusiński. `http://bs2.tugraz.at/irt-stud/pa/Literatur/Mikusinski.pdf` on 24 March 2012, unknown year.

[8] W. E. Drummond. Landau damping. *Physics of Plasmas*, 11(2):552–560, 2004.

[9] Domenico Giulini and Norbert Straumann. '...I Didn't reflect much on what I was doing...' how Planck discovered his radiation formula. *Phys.Bl.*, 2000.

[10] R. W. Gould, T. M. O'Neil, and J. H. Malmberg. Plasma wave echo. *Phys. Rev. Lett.*, 19:219–222, Jul 1967.

[11] G. Gripenberg, S.O. Londen, and O.J. Staffans. *Volterra Integral and Functional Equations*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.

[12] K.A. Jackson. *Kinetic Processes: Crystal Growth, Diffusion, and Phase Transitions in Materials*. John Wiley & Sons, 2010.

[13] L.D. Landau. Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Physikalische Zeitschrift der Sowjetunion*, 10:154–164, 1936.

[14] L.D. Landau. On the vibrations of the electronic plasma. *J.Phys.(USSR)*, 10:25–34, 1946. Reproduced in *Collected papers of L.D. Landau*, edited and with an introduction by D. ter Haar, Pergamon Press, 1965.

[15] Linge Patrick Mendele M'ekome. Le critère de penrose dans le cas périodique. Undergraduate research project report, 2011.

[16] S.G Mikhlin. *Integral equations and their applications to certain problems in mechanics, mathematical physics and technology*. Pergamon, London, 1957.

[17] Clément Mouhot and Cédric Villani. On Landau damping. *Acta Mathematica*, 207:29–201, 2011. 10.1007/s11511-011-0068-9.

[18] N.I. Muskhelishvili. *Singular integral equations:*. P. Noordhoff, 1953.

[19] R.E.A.C. Paley and N. Wiener. *Fourier transforms in the complex domain*. Colloquium Publications - American Mathematical Society. American Mathematical Society, 1934.

[20] Oliver Penrose. Electrostatic instabilities of a uniform non-maxwellian plasma. *Physics of Fluids*, 3(2):258–265, 1960.

[21] L.E. Reichl and I. Prigogine. *A modern course in statistical physics*. E. Arnold, 1980.

[22] Boris V. Somov. *Plasma Astrophysics, Part I: Fundamentals and Practice*. Astrophysics and Space Science Library. Springer, 2006.

[23] K.H. Spatschek. *Theoretische Plasmaphysik: eine Einführung*. Teubner Studienbücher: Physik. Teubner, 1990.

[24] E.M. Stein and R. Shakarchi. *Real Analysis: Measure Theory, Integration, And Hilbert Spaces*. Princeton Lectures in Analysis. Princeton University Press, 2005.

[25] E.C. Titchmarsh. *Introduction to the theory of Fourier integrals*. The Clarendon Press, 1948.

[26] András Vasy. Math 220: Duhamel's principle. `http://math.stanford.edu/~andras/220-10.pdf` on 24 March 2012, 2009.

[27] A. A. Vlasov. The vibrational properties of an electron gas. *Physics-Uspekhi*, 10(6):721–733, 1968. Reprint from Zh. Eksp. Teor. Fiz. 8, 291 (1938). Original submitted October 1, 1937.

[28] D. Werner. *Funktionalanalysis*. Springer-Lehrbuch. Springer-Verlag Berlin Heidelberg, 2007.

[29] D.V. Widder. *The Laplace transform*. Princeton mathematical series. Princeton university press, 1946. Also online available at `http://archive.org/details/laplacetransform031816mbp` (last checked May 2012).

[30] A. Y. Wong, R. W. Motley, and N. D'Angelo. Landau damping of ion acoustic waves in highly ionized plasmas. *Phys. Rev.*, 133:A436–A442, Jan 1964.